DANIEL H. WAGNER, ASSOCIATES
PAOLI OFFICE
STATION SQUARE ONE
PAOLI, PENNSYLVANIA 19301
(215) 644-3400
(800) 345-1252

H. R. RICHARDSON, MANAGER
BARRY BELKIN, TECHNICAL DIRECTOR
SCOTT S. BROWN, SR. ASSOCIATE
ROBERT P. BUEMI, SR. ASSOCIATE
LARRY K. GRAVES, SR. ASSOCIATE
WALTER R. STROMQUIST, SR. ASSOCIATE
CAROL R. HOPKINS, SR. SOFTWARE ANALYST
KATHLEEN M. SOMMAR, SR. SOFTWARE ANALYST
RICHARD H. CLARK, SOFTWARE ANALYST

HQTRS., PAOLI, PA
DANIEL H. WAGNER, PRES.

YORKTOWN OFFICE
1807 GEO. WASHINGTON HWY.
YORKTOWN, VA 23692
(804) 898-7700
JOSEPH H. DISCENZA, MGR.

SUNNYVALE OFFICE
1270 OAKMEAD PARKWAY
SUITE 314
SUNNYVALE, CA 94086
(408) 732-8393
STANLEY J. BENKOSKI, MGR.

WASHINGTON OFFICE
SUITE 111
501 CHURCH STREET, N. E.
VIENNA, VA 22180
(703) 938-2032
LAWRENCE D. STONE, MGR.
BERNARD J. McCABE, FIELD DIR.

November 15, 1984

INTERNAL MEMORANDUM

To: 450 File

From: W. R. Stromquist

Subject: Packing Unit Squares Inside Squares, III (Cases with n ≦ 65 and Martin Gardner's Conjecture for n = 11)

This memorandum is the third in a series of informal notes on the following problem: For which values of n and s can n unit squares be packed inside a square of side s? The first two memoranda, references [a] and [b], presented complete solutions in the cases with n ≦ 10.

This memorandum is in three parts. In the first section we review the best known packings in the cases of n ≦ 65, including new packings in the cases of n = 18 and n = 26. In Section 2 we establish the truth of a conjecture of Martin Gardner that n = 11 is the first case in which the best packing requires unit squares tilted at an angle other than 45°. In Section 3 we briefly cite the published asymptotic results.

1. The Best Known Packings for n ≦ 65.

Table 1 lists the side s required for the best known packings of n unit squares for selected values of n ≦ 65. For the other values of n ≦ 65, no nontrivial packings are known. (The "trivial" packings are those with $s = \lceil\sqrt{n}\rceil$; that is, the least integer $\geq \sqrt{n}$.)

TABLE 1

Best Known Packings for $n \leqq 65$

n (number of unit squares)	s (side of bounding square)	notes
2–4	2	Proved best
5	$2 + \frac{1}{2}\sqrt{2} \approx 2.707$	Proved best (reference [c])
6–9	3	Proved best (reference [a])
10	$3 + \frac{1}{2}\sqrt{2} \approx 3.707$	Proved best (reference [b])
11	≈ 3.877	Gustafsson & Thulin (reference [d], Nov. '80) (Figure 1)
17	$4 + \frac{1}{2}\sqrt{2} \approx 4.707$	Göbel (reference [c])
18	$\frac{1}{2}(7 + \sqrt{7}) \sim 4.823$	New (Figure 3)
19	$3 + \frac{4}{3}\sqrt{2} \approx 4.886$	Wainright (reference [d], Mar. '80) (Figure 2)
26	≈ 5.651	New (Figure 4)
27	$5 + \frac{1}{2}\sqrt{2} \approx 5.707$	Göbel (reference [c])
28	$3 + 2\sqrt{2} \approx 5.828$	Göbel (reference [c])
37–38	$6 + \frac{1}{2}\sqrt{2} \approx 6.707$	Göbel (reference [c])
39–40	$4 + 2\sqrt{2} \approx 6.828$	Göbel (reference [c])
50–52	$7 + \frac{1}{2}\sqrt{2} \approx 7.707$	Göbel (reference [c])
65	$5 + \frac{5}{2}\sqrt{2} \approx 8.536$	Göbel (reference [c])

Most of the results in Table 1 are due to F. Göbel, whose 1979 paper (reference [c]) was the first to deal with this problem for small n. All of Göbel's values can be justified by "45° packings," that is, packings in which all sides of the unit squares make angles of 0° or 45° with sides of the bounding square. Some improvements in Göbel's results were published by Martin Gardner in his Scientific American columns (reference [d]) for October and November 1979 and March and November 1980.

Packings with n = 11 are shown in Figure 1. Figure 1a is a 45° packing supporting Göbel's value of $s = \frac{5}{2} + \sqrt{2} \approx 3.914$, which remains the best known 45° packing. The improved packing in Figure 1b, with $s \approx 3.877$, was discovered by Mats Gustafsson and Magnus Thulin, who learned of the problem via Ronden, a publication of the Swedish pharmaceutical firm KabiVitrum. We will have more to say about the case n = 11 in Section 2.

Packings with n = 19 are shown in Figure 2. Figure 2a is by Göbel, and Figure 2b is by Charles F. Cottingham. Figure 2c, the best known packing, was discovered by Martin Gardner's readers, beginning with Robert T. Wainright. The packing in Figure 2d is not an improvement on Wainright's packing, but it seems to be the best that can be done with its theme.

Two new packings are shown in Figure 3 (n = 18) and Figure 4 (n = 26). Both are improvements on Göbel's packings, which remain the best known 45° packings in each case.

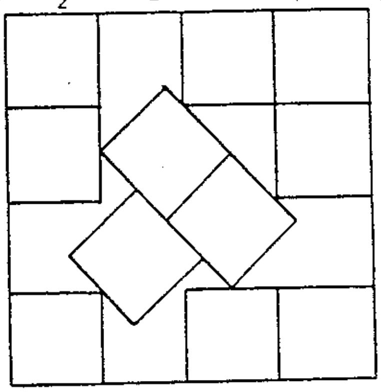
Cases with $n = k^2 + k$. Göbel was unable to improve on the trivial packings in the case of any n of the form $n = k^2 + k$. We now know that no nontrivial packings are possible in the cases of n = 2 or n = 6 (reference [a]), and it seems very unlikely that nontrivial packings will be found for n = 12 or n = 20. Nevertheless, the asymptotic results in reference [e] compel the existence of nontrivial packings with $n = k^2 + k$ for all n and k sufficiently large. What are the smallest values of n and k for which

FIGURE 1

The Case n = 11

$s = \frac{5}{2} + \sqrt{2} \approx 3.914$ (Göbel)

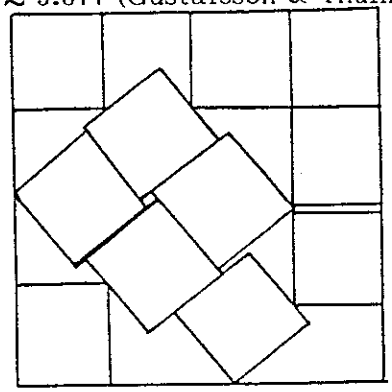
(a)



Best known 45° packing

$s \approx 3.877$ (Gustafsson & Thulin)

(b)



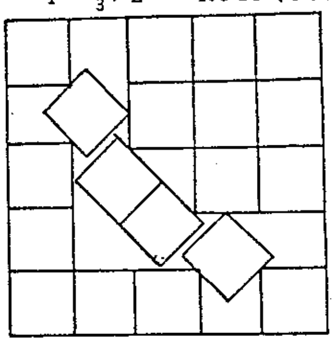
Best known packing
($\theta \approx 40.182°$)

FIGURE 2

The Case n = 19

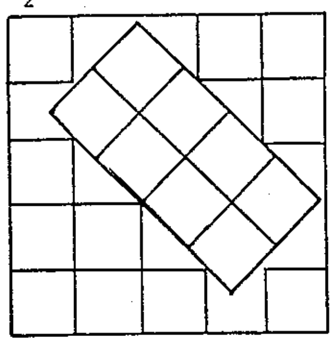
$s = 4 + \frac{2}{3}\sqrt{2} \approx 4.943$ (Göbel)

(a)



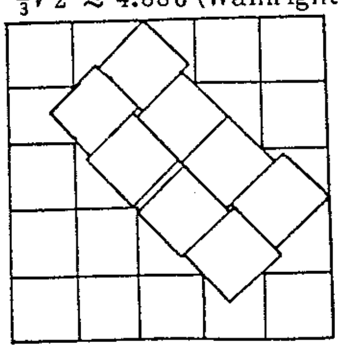
$s = \frac{5}{2} + \sqrt{2} \approx 4.914$ (Cottingham)

(b)



$3 + \frac{4}{3}\sqrt{2} \approx 4.886$ (Wainright)

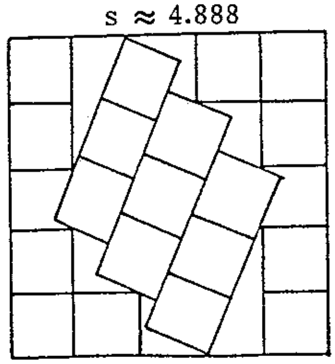
(c)



Best known packing

$s \approx 4.888$

(d)



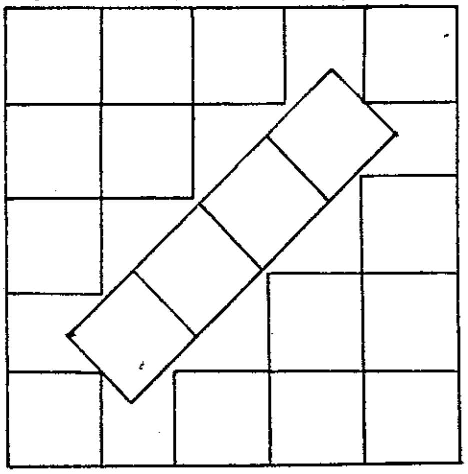
($\theta \approx 23.944°$)

FIGURE 3

The Case n = 18

$s = 2 + 2\sqrt{2} \approx 4.828$ (Göbel)

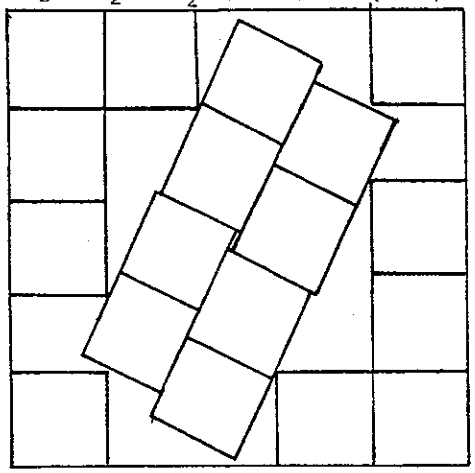
(a)



Best known 45° packing

$s = \frac{7}{2} + \frac{1}{2}\sqrt{7} \approx 4.823$ (new)

(b)



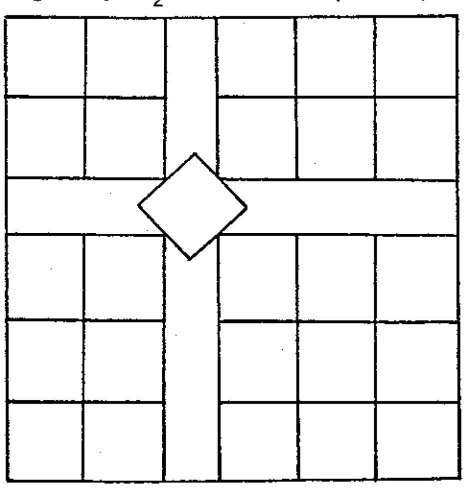
Best known packing
($\theta \approx 24.295°$)

FIGURE 4

The Case n = 26

$s = 5 + \frac{1}{2}\sqrt{2} \approx 5.707$ (Göbel)

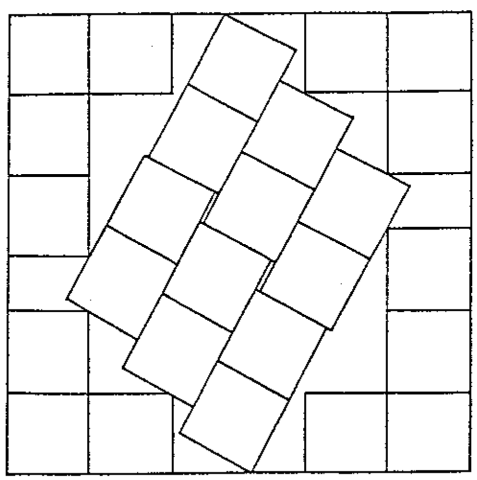
(a)



Best known 45° packing

$s \approx 5.650629$ (new)

(b)



Best known packing
($\theta \approx 27.583°$)

such packings exist? Gardner quotes Ronald L. Graham as having found a nontrivial packing with $n = 40^2 - 40 = 1560$ and suspecting the existence of nontrivial packings with n as low as $10^2 - 10 = 90$.

Lower bounds. Göbel proved that the results in Table 1 for $n \leqq 5$ are actually best possible. In references [a] and [b] we have extended this result to the cases of $n \leqq 10$.* Certain other cases ($n = 14$, $n = 15$, $n = 24$) are fairly easy to settle by elementary means, and, of course, the trivial packings are best possible when $n = k^2$. For other values of n, it is not known whether the best known packings can be improved.

One might ask for nontrivial lower bounds on the side s of a square into which n unit squares can be packed. However, except for the above mentioned cases and for very large values of n (reference [f]), no nontrivial lower bounds have been published.

It seems not to be known, even, whether there is a nontrivial packing for some n of the form $n = k^2 - 1$.

2. The Case n = 11 and Martin Gardner's Conjecture

Gardner said about the packing in Figure 1b: "This finding probably answers a question raised by Ronald L. Graham: What is the smallest number of unit squares for which the densest packing into a square requires a tilting of unit squares at any angle other than 45 degrees?"

To show that $n = 11$ is indeed the answer to this question, we need to show first that non-45° packings are not required to achieve the optimal result in cases with $n \leqq 10$. This we have done in references [a] and [b]. Second, we need to show that the 11-square packing in Figure 1b cannot be equaled or improved by a 45° packing. That is the purpose of this section. We will prove that no 45° packing is possible with $n = 11$ and $s < 2 + \frac{4}{3}\sqrt{2} \approx 3.886$.

*Göbel attributes to G. Bajmóczy of Budapest the result that no nontrivial packings exist for n = 7.

As in references [a] and [b], we define a box to be the interior of a square of side $1 + \varepsilon$, where $0 < \varepsilon < 10^{-4}$. Then the above claim is equivalent to the following theorem.

Theorem. Let $s = 2 + \frac{4}{3}\sqrt{2} \approx 3.886$. Then eleven non-intersecting boxes cannot exist inside a square of side s, if each box has orientation 0° or 45° with respect to the square.

Proof. Let the square be bounded by the axes and the lines $x = s$, $y = s$. Consider the ten points marked in Figure 5. If eleven boxes are to be packed into the square, one of them will have to avoid all of the marked points. This is impossible for a box with 0° orientation. For a 45° box, it is possible only if (up to symmetry) the box is in approximately the position shown in Figure 6. In particular, the box must contain all three of the points marked "A" in Figure 7:

$$A = \begin{cases} (s-3,\ 1) \approx (.886,\ 1) \\ (s-3,\ \frac{s}{2}) \approx (.886,\ 1.943) \\ (1.3,\ 1.5) \end{cases}$$

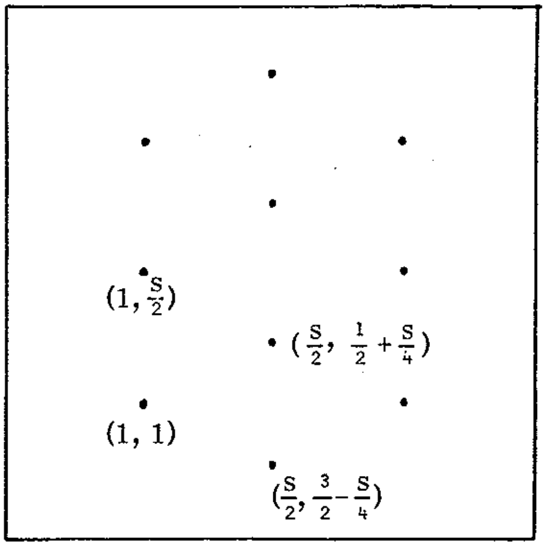
There are nine other points marked in Figure 7:

$$\begin{aligned}
B &= (1,\ s-1) \approx (1,\ 2.886) \\
C &= (\tfrac{s}{2},\ 5-.8) \approx (1.943, 3.086) \\
D &= (s-1,\ s-1) \approx (2.886,\ 2.886) \\
E &= (s-.8,\ \tfrac{s}{2}) \approx (3.086,\ 1.943) \\
F &= (s-1,\ 1) \approx (2.886,\ 1) \\
G &= (s-2,\ .8) \approx (1.886,\ .8) \\
H &= (1.7,\ 2.2) \\
I &= (2.2,\ 2.2) \\
J &= (2.2,\ 1.7)
\end{aligned}$$

(Note: Figure 8 repeats Figure 7, with certain distances marked. Each of these distances is less than 1.)

FIGURE 5

TEN POINTS TO AVOID . . .



Points not labeled are placed symmetrically.

FIGURE 6

. . . AND HOW TO AVOID THEM

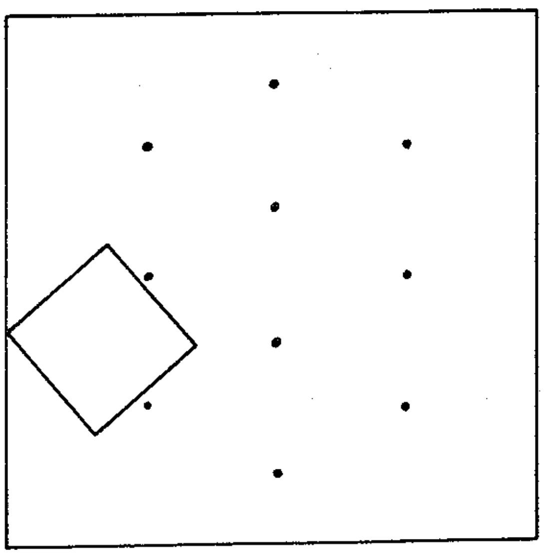


FIGURE 7

POINTS USED TO PROVE THE THEOREM

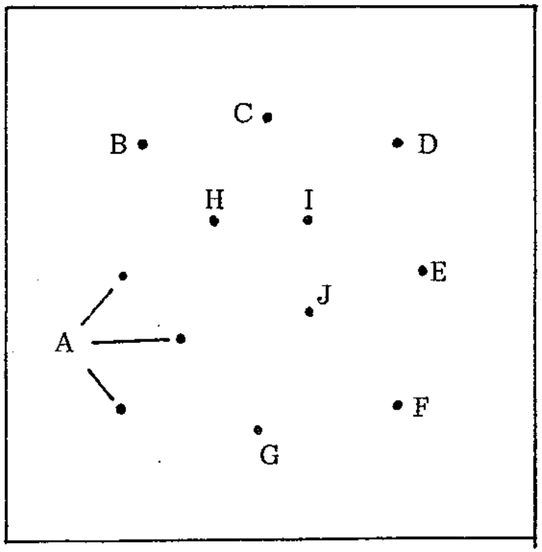
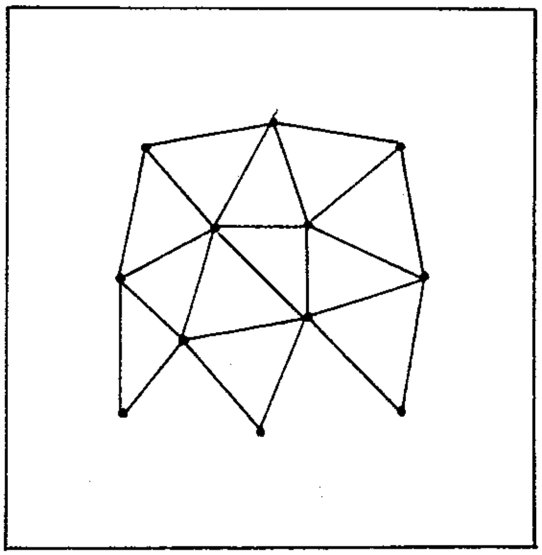


FIGURE 8

CONNECTED POINTS HAVE DISTANCE < 1



It is not hard to prove that any 0° or 45° box inside the square must contain one of the 12 points marked in Figure 7. But since three of the points are in one box, there cannot be eleven non-overlapping boxes. This completes the proof of the theorem and establishes the truth of Martin Gardner's conjecture.

(By reducing the value of s, essentially the same argument can be made to work for general packings. The result is that 11 unit squares cannot be packed inside a square with side $s < 2 + \frac{4}{5}\sqrt{5} \approx 3.789$.)

A digression: It may seem that non-45° packings are an unusual phenomenon when packing a small number of squares into squares. But in the related problem of packing unit squares into rectangles, non-45° packings appear much more quickly. For example, four unit squares can be packed into a rectangle with sides 1.9 and 3.95, but only by a non-45° packing. An optimal packing of this sort is shown in Figure 9. Here $\theta \approx 39.63°$ satisfies

$$\cos^3\theta - \sin^3\theta = .95\cos^2\theta - .90\sin^2\theta,$$

and the rectangle has dimensions 1.9 and 3.9475^+.

3. The Asymptotic Results

The purpose of this section is just to mention the published asymptotic results, references [e] and [f]. The author knows of no published papers dealing with packing unit squares in squares other than references [c] through [f].

In reference [e], Erdős and Graham define

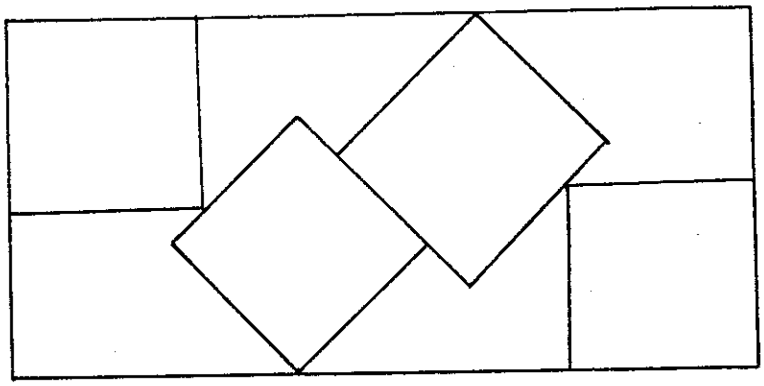
$$W(s) = s^2 - n_{max}$$

where n_{max} is the largest n such that n unit squares can be packed into a square of side s. Thus, W(s) is the wasted area in the optimal packing with side s. They show

FIGURE 9

AN OPTIMAL RECTANGLE PACKING

$s_1 = 1.9, s_2 \approx 3.9475$



4 unit squares ($\theta \approx 39.63°$)

(by an explicit packing) that

$$W(s) = O(s^{7/11}).$$

Montgomery (in an unpublished argument mentioned in reference [f]) has strengthened the result to

$$W(s) = O\left(s^{\frac{3}{2} - \frac{1}{2}\sqrt{3} + \varepsilon}\right)$$

where the exponent is approximately 0.634 and ε is positive. In reference [f], Roth and Vaughan establish a nontrivial lower bound for W(s) (provided $s(s - [s]) > \frac{1}{6}$):

$$W(s) \geqq 10^{-100} \sqrt{\| s \| s}$$

where $\| s \|$ is the difference between s and the nearest integer to s. This implies that Erdös and Graham's exponent could not be reduced below $\frac{1}{2}$.

If there is to be a fourth memorandum in this series, it will deal either with the asymptotic results or with computerized methods for finding good packings for moderate values of n.

Walter Stromquist

Walter R. Stromquist

WRS/hh

References

[a] Packing Unit Squares Inside Squares, I (Six Unit Squares), Daniel H. Wagner, Associates Internal Memorandum to 450 File, W. R. Stromquist, September 11, 1984.

[b] Packing Unit Squares Inside Squares, II (Ten Unit Squares), Daniel H. Wagner, Associates Internal Memorandum to 450 File, W. R. Stromquist, October 15, 1984.

[c] "Geometrical Packing and Covering Problems," by F. Göbel, Mathematical Centre Tracts, 1979.

[d] "Mathematical Games," by M. Gardner, Scientific American, October 1979 (see also November 1979, March 1980, and November 1980).

[e] "On Packing Squares with Equal Squares," by P. Erdös and R. L. Graham, Journal of Combinatorial Theory, 19, 1975.

[f] "Inefficiency in Packing Squares with Unit Squares," by K. F. Roth and R. C. Vaughan, Journal of Combinatorial Theory, Series A 24, 1978.