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October 15, 1984

INTERNAL MEMORANDUM

To: 450 File (Professional Leave)

From: W. R. Stromquist

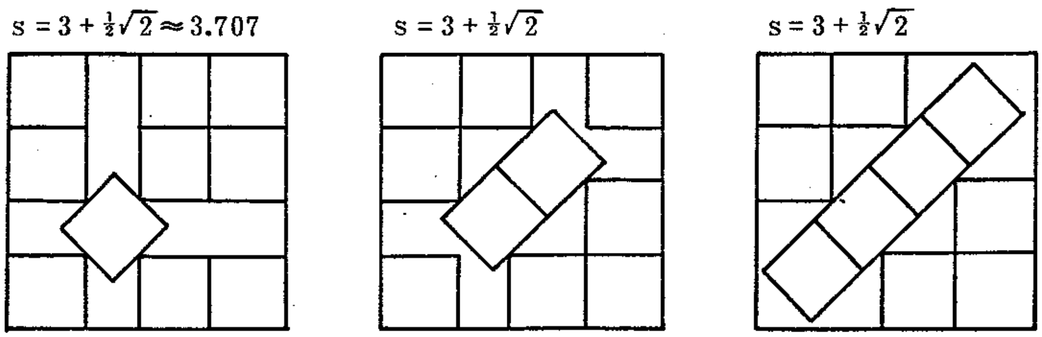
Subject: Packing Unit Squares Inside Squares, II (Ten Unit Squares)

This memorandum is the second of a series on the following general problem: For which values of n and s can n unit squares be packed inside a square of side s? The first memorandum, reference [a], dealt primarily with the case of $n = 6$, and also settled the remaining cases with $n \leq 9$. This memorandum addresses the case of $n = 10$. We will prove that ten unit squares can be packed inside a square of side $s = 3 + \frac{1}{2}\sqrt{2} \approx 3.707$, but not inside any smaller square.

Three different packings of ten unit squares in a square of side $s = 3 + \frac{1}{2}\sqrt{2}$ are shown in Figure 1.

FIGURE 1

THREE PACKINGS OF TEN UNIT SQUARES



We will show that no packing of ten unit squares is possible if the bounding square has side less than $3 + \frac{1}{2}\sqrt{2}$. Actually, we will prove an equivalent result, that ten squares cannot be packed inside a square of side exactly $3 + \frac{1}{2}\sqrt{2}$, if each of the smaller squares has side strictly greater than 1. Following reference [a], we define a block to be the interior of any unit square of side $(1 + \varepsilon)$, where $0 < \varepsilon < 10^{-4}$. We will prove the following:

Theorem. Ten pairwise nonintersecting blocks cannot exist in the interior of a square of side $s = 3 + \frac{1}{2}\sqrt{2}$.

The proof will be in Section 2, and will make use of lemmas presented in Section 1.

1. Some Nonavoidance Lemmas

The four lemmas presented in this section are what are referred to in reference [a] as "nonavoidance lemmas." Each lemma provides that if the center of a block is in a certain region, then the block must have a nonempty intersection with certain parts of the boundary of the region. All four lemmas are illustrated in Figure 2.

The first two lemmas are proved in reference [a].

Lemma 1. Consider the square bounded by the lines $x = 0$, $x = 1$, $y = 0$, $y = 1$. Any block whose center lies on or inside this square, and which does not intersect either the x axis or the y axis, must contain the point $(1,1)$. (See Figure 2a).

This is corollary 2 to lemma 1 of reference [a]. Clearly the lemma remains valid if the lines $x = 1$, $y = 1$ are replaced by $x = a$, $y = b$, and the point $(1,1)$ is replaced by (a,b), provided $a,b \leq 1$.

Lemma 2. Consider a triangle whose sides each have length at most 1. Any block whose center lies on or inside the triangle must contain one of the vertices of the triangle. (See Figure 2b.)

This is lemma 2 of reference [a].

FIGURE 2

NONAVOIDANCE LEMMAS

Note: In each case, if the center of a block is in the shaded region, the block must intersect one of the marked lines or points.

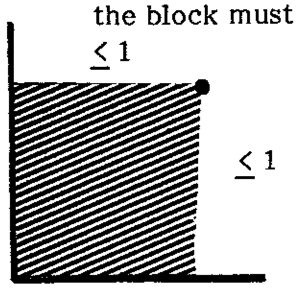


Figure 2a
(Lemma 1)

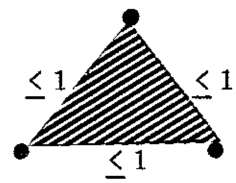


Figure 2b
(Lemma 2)

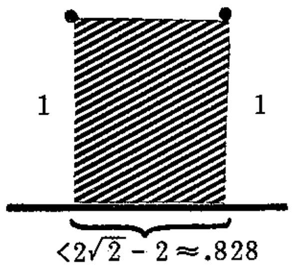


Figure 2c
(Lemma 3)

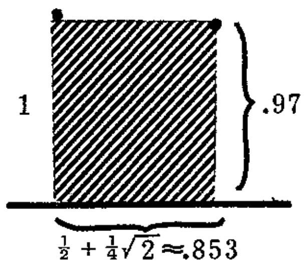


Figure 2d
(Lemma 3)

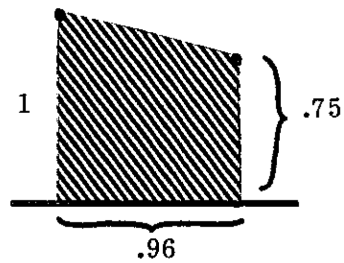


Figure 2e
(Lemma 3)

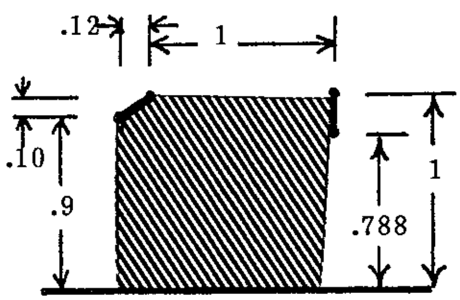


Figure 2f
(Lemma 4)

Lemma 3. Consider the quadrilateral R with vertices at (0,0), (0,1), (a,0), (a,b). Suppose that

(1) $a = 2\sqrt{2} - 2 \approx .828$ and $b = 1$, or

(2) $a = \frac{1}{2} + \frac{1}{4}\sqrt{2} \approx .853$ and $b = .97$, or

(3) $a = .96$ and $b = .75$.

Then any block whose center lies on or inside R, and which does not intersect the x axis, must contain one of the vertices (0,1) and (a,b). (See Figures 2c, 2d, and 2e.)

Proof. Part (1) is from reference [a], but an independent proof of all three parts will be given here.

Let A = (0,1), and B = (a,b). In each case, the distance between A and B is less than 1; therefore, any counterexample could not have two corners above the line AB. By lemma 1, any counterexample must have one corner to the left of x = 0, and one corner to the right of x = a. If there is a counterexample, then without loss of generality it has a vertex on the x axis and has the point A on its boundary, as in Figure 3. Let θ be the angle between the boundary of the block and the x axis, as shown in the figure.

Suppose that the block's upper boundary intersects the line x = a at the point (a,y). We will show that even if θ is chosen to minimize y, we have $y > b$, showing that no counterexample is possible.

Figure 4 shows the derivation of the formula for y in terms of a and θ:

$$y > f(\theta) = 1 - \frac{1}{1+\cos\theta} + \frac{1-a\cos\theta}{\sin\theta} .$$

FIGURE 3

GENERAL FORM OF ANY COUNTEREXAMPLE TO LEMMA 3

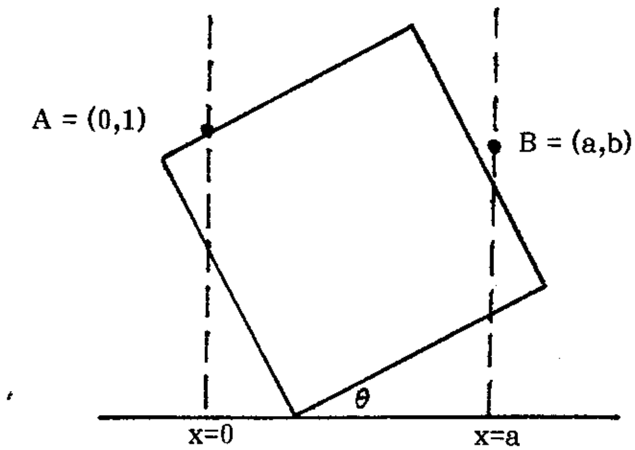
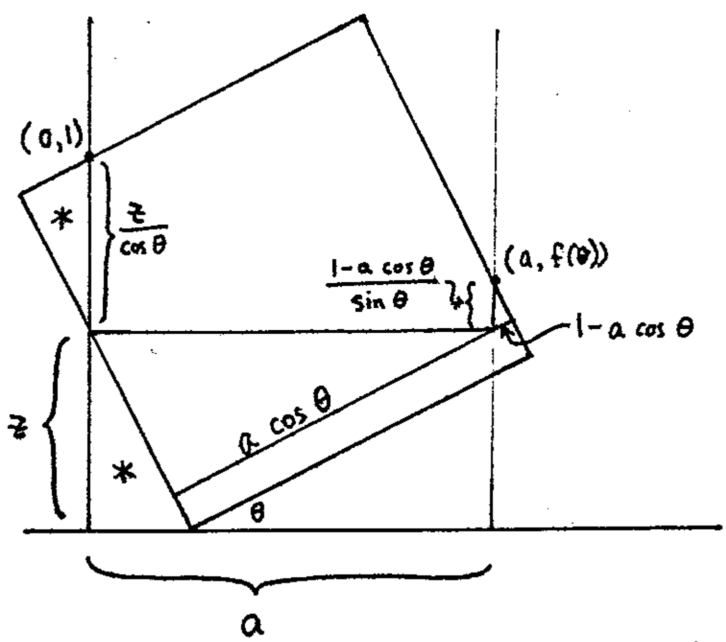


FIGURE 4

DERIVATION OF f(θ) IN PROOF OF LEMMA 3



Note: The two triangles marked * are congruent. Since $z + \frac{z}{\cos\theta} = 1$,

we have

$$z = 1 - \frac{1}{1+\cos\theta},$$

so that

$$f(\theta) = 1 - \frac{1}{1+\cos\theta} + \frac{1-a\cos\theta}{\sin\theta}.$$

(We would have $y = f(\theta)$ if we were dealing with a unit square, but since the block is larger than a unit square, we have $y > f(\theta)$.) To minimize $f(\theta)$, we set its derivative equal to zero:

$$f'(\theta) = \frac{(a-\cos\theta)(1+\cos\theta) - \sin\theta\,(1-\cos\theta)}{\sin^2\theta\,(1+\cos\theta)} = 0. \qquad (1)$$

When $2\sqrt{2} - 2 \leq a < 1$, equation (1) has two roots in [0°, 90°], and the smaller of them (with $\theta \leq 45°$) is a local minimum for $f(\theta)$. (In the interval [0,90°], $f(\theta)$ also has a local minimum at $\theta = 90°$, when $f(\theta) = 1 > b$.)

The relevant root of $f'(\theta)$ can be found either by solving a cubic in $(\cos\theta)$ or by search. The following table shows the root θ, and the value of $f(\theta)$, for the relevant values of a.

	a	θ = smaller root of eq. (1)	$f(\theta)$	b
(1)	$2\sqrt{2} - 2 \approx .828$	45°	1	1
(2)	$\frac{1}{4}\sqrt{2} + \frac{1}{2} \approx .853$	39.514°	.9722	.97
(3)	.96	17.708°	.7689	.75

Since $f(\theta) \geq b$ in each case, the lemma is proved.

Lemma 4. Consider the pentagon with vertices at (.88, 0), (.88, .90), (1,1), (2,1), and (2,0). Any block whose center lies on or inside this pentagon must intersect either

(a) the x axis, or

(b) the segment joining (.88, .90) and (1,1), or

(c) the segment joining (2, 1) and (2, .788).

Proof. By lemma 1, any counterexample would need to include a point to the right of the line x = 2. Without loss of generality, the block's boundary includes a point on the x axis and the point (2, .788). See Figure 5. Let θ be the angle with the x axis, as shown.

If $\theta \geq \tan^{-1}(5/6) \approx 39.8°$, then the block must contain the point (1,1). To see this, we estimate the x-coordinate of the point (x,1) at which the block's left boundary intersects the line y = 1. As shown in Figure 6, x is given by

$$x = 2 - .212 \tan\theta - \frac{\sin\theta + \cos\theta - 1}{\sin\theta \cdot \cos\theta} \tag{2}$$

The last term is derived in reference [a], where its absolute value is shown to be at least .828, whatever the value of θ. The next-to-last term in (2) must have absolute value at least (.212) (5/6) > .176. It follows that x < 2 - .176 - .828 = .996, so that the block's boundary passes to the left of (1,1) as claimed.

If $\theta < \tan^{-1}(5/6)$ but $\theta \geq \cos^{-1}(.9) \approx 25.8°$, then the block must contain the point (.88, .90). To see this, we compute the x-coordinate of the point (x, .90) at which the block's left boundary intersects the line y = .90. By the same method as in the previous paragraph, we obtain

FIGURE 5

GENERAL FORM OF ANY COUNTEREXAMPLE TO LEMMA 4

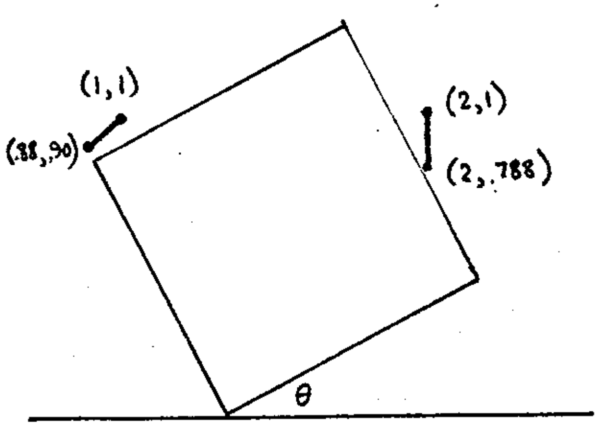
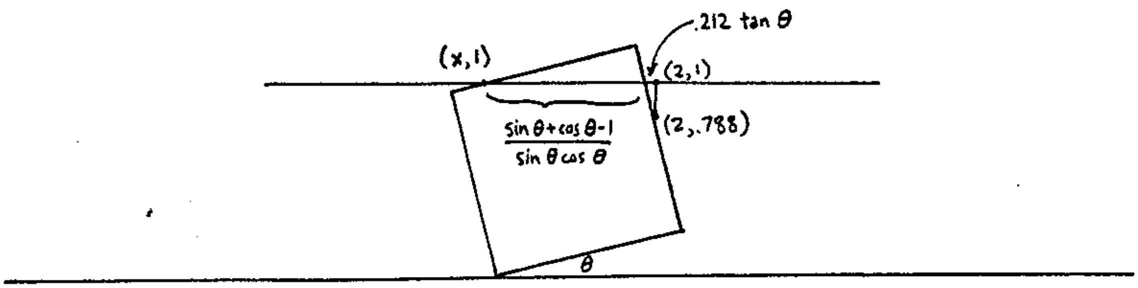


FIGURE 6

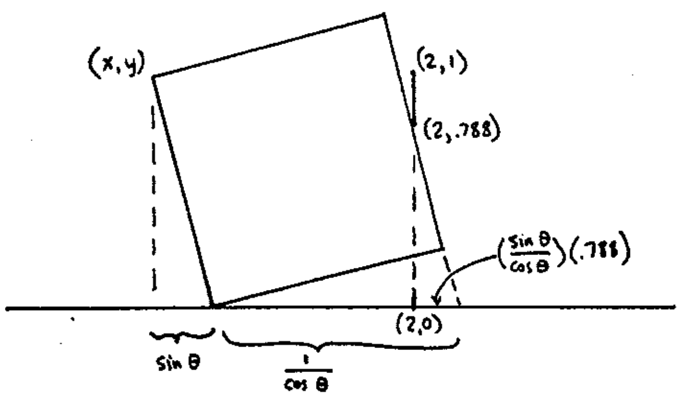
PROOF OF LEMMA 4: INTERSECTION OF BLOCK WITH LINE y = 1



$$x = 2 - .212 \tan\theta - \frac{\sin\theta + \cos\theta - 1}{\sin\theta + \cos\theta}$$

FIGURE 7

CALCULATION OF x-COORDINATE IN PROOF OF LEMMA 4



$$x = 2 + \frac{\sin\theta}{\cos\theta}(.788) - \frac{1}{\cos\theta} - \sin\theta$$

$$x = 2 - .112 \tan \theta - \frac{\sin \theta + \cos \theta - .90}{\sin \theta \cos \theta} .$$

This function of θ reaches its maximum at $\theta = 37.40°$, when $x = .87441$. Thus, the block's left boundary passes to the left of (.88, .90) as claimed.

If $\theta < \cos^{-1}(.9)$, we need to calculate the coordinates (x,y) of the block's leftmost vertex. We have:

$$y = \cos \theta, \text{ and}$$
$$x = 2 + \frac{.788 \sin \theta}{\cos \theta} - \frac{1}{\cos \theta} - \sin \theta$$

where the last formula is derived in Figure 7. Since $.9 < y < 1$, the vertex will be to the left of the segment if $(1-x)/(1-y) > 1.2$. We calculate:

$$\frac{1-x}{1-y} = \frac{\sin \theta - 1 + \dfrac{1 - .788 \sin \theta}{\cos \theta}}{1 - \cos \theta}$$

If $\theta \leq 20°$, we have (since $\cos \theta \leq 1$):

$$\frac{1-x}{1-y} \geq \frac{\sin \theta - 1 + (1 - .788 \sin \theta)}{1 - \cos \theta}$$

$$= (.212) \frac{\sin \theta}{1 - \cos \theta}$$

$$= (.212) \frac{1 + \cos \theta}{\sin \theta} \geq (.212) \frac{1 + \cos 20°}{\sin 20°} > 1.202.$$

If $20° < \theta < \cos^{-1}(.9)$, simpler estimating techniques establish that $\frac{1 - x}{1 - y}$ is always greater than 1.6. Thus, the block has a vertex directly to the left of the segment, and must intersect the segment. This completes the proof of lemma 4.

2. Proof of the Theorem

In this section, s always represents the number $3 + \frac{1}{2}\sqrt{2}$. Consider the square S bounded by the axes and the lines x = s, y = s. We will suppose that ten nonoverlapping blocks are contained in the interior of S, and argue to a contradiction.

Consider the ten points A, B, ..., J marked in Figure 8. As shown in the figure, the coordinates of A, B, J are A = (1, 1), B = $(\frac{s}{2}, .97)$, J = $(\frac{s}{2}, 1.4)$; the other points are placed symmetrically in the square. We claim that any block contained in S must contain one of these ten points. The proof is in Figure 9, where S is divided into regions, to each of which Lemma 1, 2, or 3 applies.

Since there are ten blocks and ten of these points, each block must contain exactly one of the points. We will refer to the blocks by the points they contain; e.g., the A-block, the B-block, etc.

The above reasoning would work just as well if I and J were replaced by the points U = (1.4, s/2) and V = (s - 1.4, s/2) (Figure 10). Clearly points U and V are contained in the I-block and J-block (in either order).

The same reasoning would also work if the point B in Figure 8 were replaced by the point W = (s - 1.96, 0.75). The proof is as in Figure 9, the only difference being that part 3 of Lemma 3 applies, rather than part 2. Therefore, the point W must be contained in the B-block.

Now consider the eleven points marked in Figure 11: that is, points C through J, point W, and the points (1, 1.2) and (.788, 1). By an argument similar to Figure 9, every block must contain one of these eleven points. The A-block, therefore, must contain one of the last two points.

FIGURE 8

TEN POINTS A,...,J

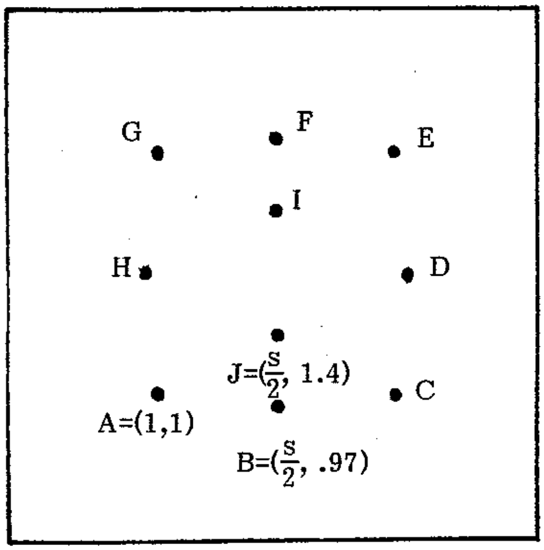


FIGURE 9

EACH BLOCK CONTAINS ONE OF A,...,J

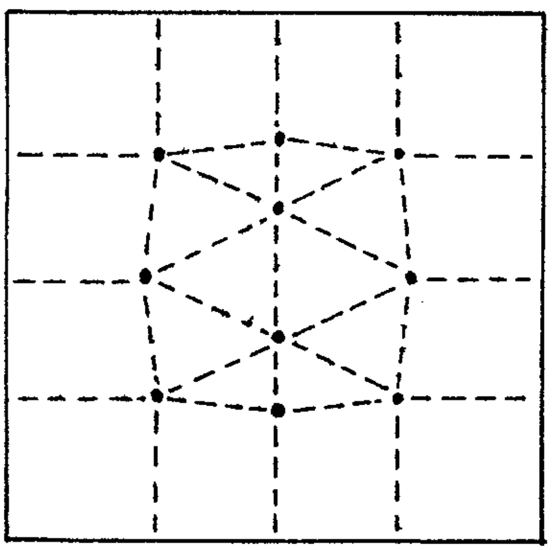


FIGURE 10

POINTS U, V ARE CONTAINED IN THE I-,J-BLOCKS

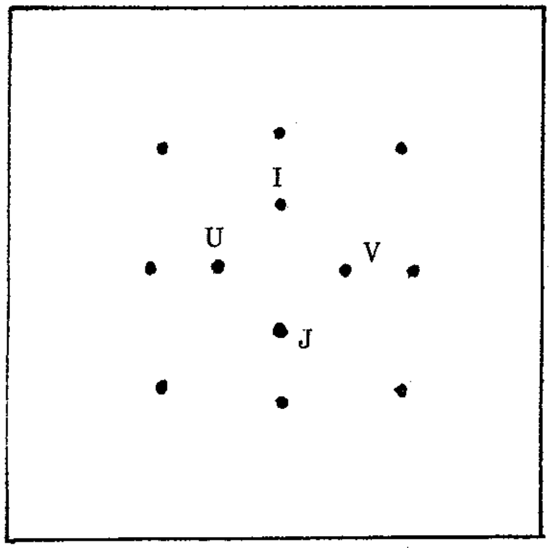
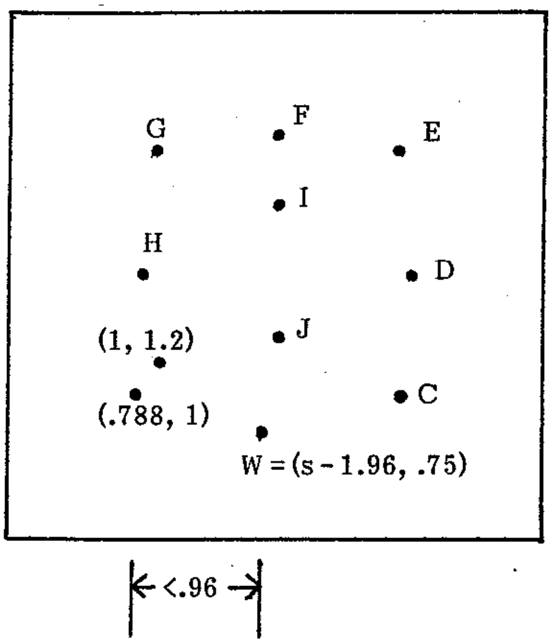


FIGURE 11

EVERY BLOCK CONTAINS ONE OF THESE ELEVEN POINTS



We now assert that the H-block must contain a point on the segment from (1,2) to (.9, 2.12). This is easy to see if the A-block contains (1, 1.2); in that case the H-block must contain (1,2). If the A-block contains (.788, 1), then it contains the entire segment from (.788, 1) to (1, 1) and lemma 4 applies, as shown in Figure 12.

In either case, the H-block must contain some point of the indicated segment. In Figure 13, the point of intersection is marked with an asterisk. Seven other asterisks mark other points which must be contained in the B-, D-, F-, and H-blocks by symmetrical arguments. Although we do not know the exact positions of

FIGURE 12

THE H-BLOCK MUST CONTAIN A POINT ON THE INDICATED SEGMENT

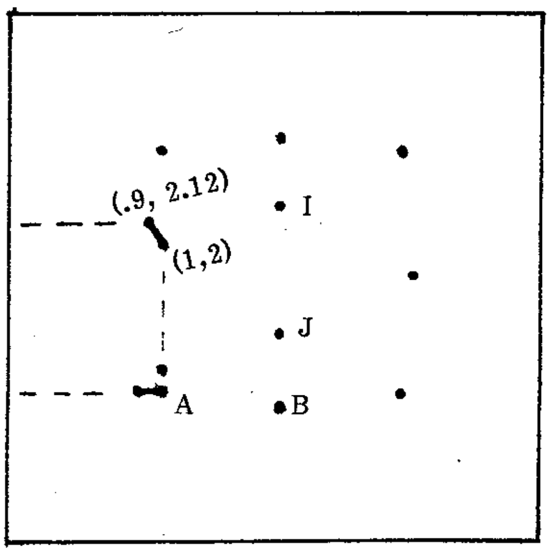


FIGURE 13

EIGHT POINTS WHICH MUST BE CONTAINED IN THE B-, D-, F-, AND H-BLOCKS

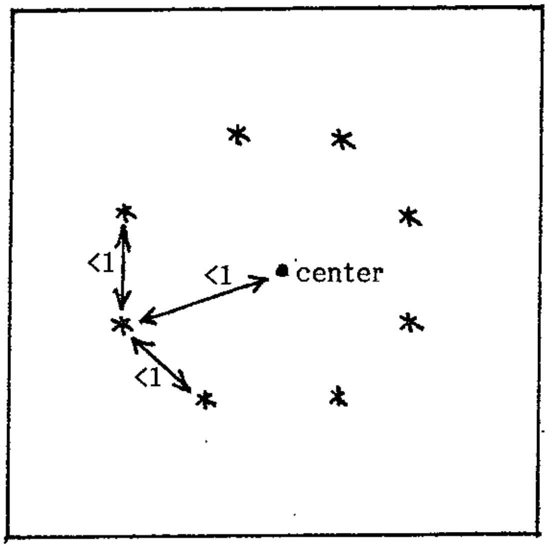
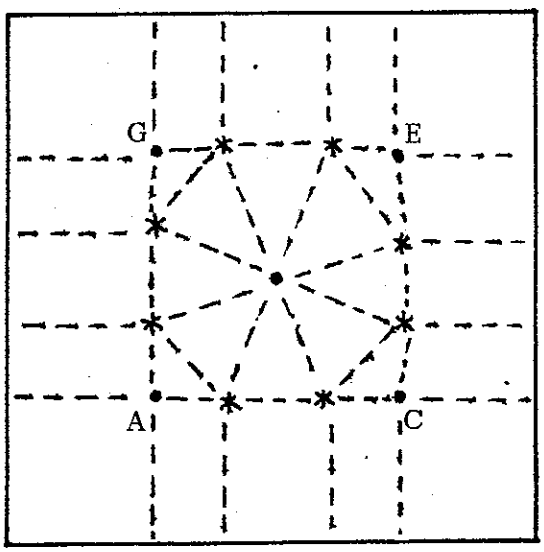


FIGURE 14

EVERY BLOCK MUST CONTAIN ONE OF THESE THIRTEEN POINTS, BUT EACH IS DENIED TO THE I-BLOCK



these points, we know that each asterisk is within 1 of the center of the square, and within 1 of each of the two asterisks nearest to it.

Now, of the I- and J-blocks, both cannot contain the center of the square. Without loss of generality, assume that the I-block does not contain the center. Consider the thirteen points marked in Figure 14: these are the eight asterisks, the four corner points A, C, E, G, and the center of the square. An application of the lemmas as shown in the figure shows that every block must contain one of these thirteen points, but all are denied to the I-block, and that is a contradiction, which completes the proof of the theorem.

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Walter R. Stromquist

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Reference

[a] Packing Unit Squares Inside Squares I (Six Unit Squares), Daniel H. Wagner, Associates Internal Memorandum to 450 File by W. R. Stromquist, September 11, 1984.