Daniel H. Wagner Associates
INCORPORATED
CONSULTANTS
OPERATIONS RESEARCH • MATHEMATICS • SOFTWARE DEVELOPMENT

HQTRS AND PENNSYLVANIA OFFICE
SUITE 200
40 LLOYD AVENUE
MALVERN, PA 19355
610 644-3400
FAX: 610 644-6293

WASHINGTON OFFICE
SUITE 206
450 MAPLE AVENUE, EAST
VIENNA, VA 22180
703 938-2032
FAX: 703 255-4781

HAMPTON OFFICE
SUITE 500
2 EATON STREET
HAMPTON, VA 23669
804 727-7700
FAX: 804 722-0249

SANTA CLARA OFFICE
SUITE 400
4677 OLD IRONSIDES DRIVE
SANTA CLARA, CA 95054-1826
408 987-0600
FAX: 408 987-0606

April 24, 1996

MEMORANDUM

Originating Office: PA
(6510)

To: Larry Forest

From: Walter Stromquist

Subject: Roots of Transition Matrices

An order-n *transition matrix* $A = (a_{ij})$ is an $n \times n$ square matrix with non-negative real entries whose row sums are all equal to 1; that is, $\sum_j a_{ij} = 1$ for each i. For example, if we classify businesses into risk grades indexed by $i = 1, \ldots, n$, then a_{ij} might represent the fraction of businesses in risk grade i that have risk grade j one year later. If we have such a matrix, we might want to produce similar statistics describing the transitions that occur during one month. If we assume that the transition processes during the twelve months of the year are identical and independent, then we are asking for a transition matrix B such that the power B^{12} is equal to A. Equivalently, we want to evaluate $A^{1/12}$.

This memorandum addresses the problem of computing an arbitrary positive power A^p of a transition matrix A, when it exists.

If we restrict ourselves to matrices with real entries, then some fractional powers may not exist. For example, the matrix

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

cannot have a square root, since one of the eigenvalues of J is -1, and that would have to be the square of one of the eigenvalues of the square root. But if we further restrict ourselves to matrices A that are sufficiently close to the identity, and in particular have positive diagonal entries, then we will find that arbitrary positive powers exist and are unique. Fractional powers (roots) may fail to be transition matrices, because they may contain negative entries.

The key to the computation is the binomial formula

$$(I + X)^p = I + \binom{p}{1} X + \binom{p}{2} X^2 + \binom{p}{3} X^3 + \cdots . \tag{1}$$

If we apply this formula with $I + X = A$, we have the required powers provided the series converges. A sufficient condition for convergence is that each entry in X be less than $1/n$ in absolute value. The series is likely to converge for transition matrices we meet in practice even if a few terms violate this condition.

(The binomial coefficients $\binom{p}{k}$ in equation (1) are unusual in that the upper entry need not be an integer. The definition we use for these binomial coefficients is

$$\binom{p}{k} = \frac{(p)(p-1)(p-2)\cdots(p-k+1)}{k!} \tag{2}$$

with k factors in each of the numerator and denominator. If p is an integer, then this expression vanishes whenever $k > p$ and the series in equation (1) is actually a finite sum. If p is not an integer, then $\binom{p}{k}$ is never zero and the series is infinite.)

Equation (1) always gives a matrix with row sums equal to one, but it does not guarantee non-negative entries. For example, we can compute

$$\begin{pmatrix} .81 & .19 & 0 \\ 0 & .81 & .19 \\ 0 & 0 & 1 \end{pmatrix}^{1/2} = \begin{pmatrix} .9 & \frac{19}{180} & -\frac{1}{180} \\ 0 & .9 & .1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

This is the only square root with positive diagonal entries, but because the square root has a negative entry off the diagonal, it cannot be a transition matrix. This means that the first matrix in the example cannot arise as a one year transition matrix resulting from two identical and independent half-year transition processes.

We'll come back to that problem, but first let's explore some more powerful series methods for the computation. We can define the logarithm of a transition matrix by

$$\log A = \lim_{p \to 0^+} \frac{1}{p} \left(A^p - I \right). \tag{4}$$

It follows from equations (4) and (1) that the logarithm is given by

$$\log(I + X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{1}{4}X^4 + \cdots \tag{5}$$

which converges whenever (1) converges for small p; that is, whenever $I + X$ is sufficiently close to the identity. We can also define an exponential function for matrices, by

$$\exp Y = I + Y + \frac{1}{2!}Y^2 + \frac{1}{3!}Y^3 + \cdots \tag{6}$$

which always converges and is an inverse function for the logarithm wherever the logarithm is defined. Now the p-th power of A can be computed by

$$A^p = \exp(p \log A). \tag{7}$$

If the off-diagonal entries of $\log(A)$ are all non-negative, then positive powers A^p for $p > 0$ have all non-negative entries and qualify as transition matrices. The matrices A

with this property are the only ones that can arise as transition matrices from an infinitely divisible process; that is, the only ones that can arise if the process results from identical and independent processes on arbitrarily small intervals. We refer to a matrix with this property as an *infinitely divisible* transition matrix. (In other words: for matrices A with logarithms, A is infinitely divisible if and only if $A = \exp Y$ for some matrix Y with non-negative off-diagonal entries.)

The off-diagonal entries of $\log A$ are the instantaneous transition rates that are consistent with a one-year transition matrix A. For example, if the 12-entry of $Y = \log A$ is y_{12}, that means that the instantaneous rate flow of companies from grade 1 to grade 2 is y_{12} per year times the number of companies currently in grade 1. An instantaneous transition model makes sense in this context only if the rates are all non-negative. If they are, then the instaneous rates determine the tranistion matrices for all discrete time intervals.

So here is one way to deal with a one-year transition matrix A:

1. Hope that A is infinitely divisible;
2. Compute monthly, quarterly, or other short-period transition matrices from (1) or (7).

What should we do if we are given a one-year transition matrix which is not infinitely divisible? This will be the case, for example, of any ordinary transition matrix that includes a zero probability for default within a year by a AAA-rated borrower. (It is easy to understand why such defaults do not occur within a given sample period, but it is absurd to argue that such defaults cannot occur at all. Usually, if we are given an empirically-derived transition matrix with a zero entry we should recognize that it is an approximation to the true matrix, and that the true matrix has a small positive entry.)

I propose that when the one-year transition matrix A is not infinitely divisible, we proceed as follows:

1. Compute $Y = \log A$.
2. Revise Y to remove the negative off-diagonal entries. Call the result Y'.
3. Replace A for all purposes with $A' = \exp Y'$.

The logic here is that a non-infinitely-divisible transition matrix is inconsistent with our model, and should be regarded as an approximation to a better transition matrix that is infinitely divisible.

Kathy Sommar has carried out this program with the 19×19 transition matrix presented as the one-year grade transition matrix in the GMW paper. She verified that the row sums of A were equal to 1 (within ordinary rounding errors) and computed its logarithm L, whose row sums were equal to 0. The matrix L did contain negative off-diagonal entries, but the most extreme of those values was -0.000006 (in units of fraction per year)—barely larger than rounding errors. She replaced the negative values by zeros, adjusted the diagonal entries in L to preserve the row sums, and exponentiated the result to produce a "corrected" version A'. The three attached pages show the initial matrix A, the unrevised logarithm L, and the corrected transition matrix A' (labeled "exponentiated A"). All entries are multiplied by 100 for readability. As you can see, the corrected version A' is essentially indistinguishable from the original. All positive powers of the corrected matrix are readily computable transition matrices.

If the negative entries had been more substantial, we would have had to think harder about how to revise the logarithm.

Walter Stromquist

(attachments: three pages)