A Pie That Can't Be Cut Fairly

Walter Stromquist

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Abstract

David Gale asked in [8, 1993] whether, when a pie is to be divided among n claimants, it is always possible to find a division that is both envy free and undominated. The pie is cut along n radii and the claimants' preferences are described by separate measures.

We answer Gale's question in the negative for $n = 3$ by exhibiting three measures on a pie such that, when players have these measures, no division of the pie can be both envy free and undominated. The measures assign positive values to pieces with positive area.

1 Cutting pies

Mathematicians study cake cutting as a metaphor for more general fair division problems. Early papers on cake cutting include [7] and [10], and many more references can be found in the recent books by Brams and Taylor [5], Robertson and Webb [9], and Barbanel [1]. A recent survey is by Brams [2].

A cake is cut by planes parallel to a given plane. It can be represented as an interval $[0, m]$ with possible cuts corresponding to points in the interval and possible pieces to subintervals $[a, b] \subseteq [0, m]$. We assume n players (claimants) whose preferences are represented by nonatomic measures on the interval. Letting v_i stand for the i-th player's measure, we write $v_i(S)$ or just $v_i(a, b)$ for the i-th player's valuation of the piece $S = [a, b]$. We always assume that $v_i(S) = 0$ if S has length 0. By an *allocation* we mean a partition of the pie into n pieces together with an assignment of pieces to players.

An allocation is called *envy free* if $v_i(S_i) \geq v_i(S_j)$ for every i and j, where S_i represents the i-th player's piece. This means that no player prefers another player's piece to its own. A player i is *satisfied* (non-envious) if $v_i(S_i) \geq v_i(S_j)$ for all j, so that an allocation is envy free precisely when all players are satisfied. An allocation $\{S_i\}$ is *dominated* by another allocation $\{T_i\}$ if $v_i(T_i) \geq v_i(S_i)$ for each i with strict inequality for at least one i. This means that the allocation $\{T_i\}$ makes at least one player better off without making any player worse off. We say that an allocation is *undominated* (or, synonymously, *efficient* or *Pareto optimal*) if it is not dominated by any other allocation.

In this paper we are looking for allocations that are both envy free and undominated. Note that each of these properties is defined without interpersonal comparisons of values.

An early result for cakes [10] is that given n players with nonatomic measures, it is always possible to find an envy-free allocation using only $n-1$ cuts. Gale showed in [8] that an envy-free allocation with $n-1$ cuts is necessarily undominated. (This result requires the additional assumption that the measures are absolutely continuous with respect to each other; that is, for each i and j, $v_i(S) > 0$ whenever $v_j(S) > 0$.) The lesson of this cake-cutting model, then, is that envy-free, undominated divisions are always possible.

Pies are an alternative metaphor. Pies differ from cakes in that they are cut along radii; that is, along straight lines from the center. While we can still represent a pie by an interval $[0, m]$, we now consider the endpoints to be identified, so that the interval represents a circle that we may visualize as the circumference of the pie. A piece of pie is represented by a subinterval $[a, b]$, but we now interpret the endpoints *mod* m; that is, $a + m$ represents the same point as a. We allow the notation $[a, b]$ with $b < a$, interpreting it as $[a, b + m]$; this typically represents a piece that spans the endpoints and would not be a single piece if it were part of a cake. The notation $[a, a]$ always represents an empty piece, while $[0, m]$ always represents the whole pie. We continue to represent the players' preferences by nonatomic measures v_i.

There still exist envy-free allocations. In fact, there are more of them for pies than for cakes, because we can cut the pie at any point and then find an envy-free allocation as if we were dividing a cake. But when $n \geq 3$ it is no longer the case that all envy-free allocations are undominated. Gale asked whether there must exist an allocation that is both envy free and undominated. Brams and Barbanel [3] showed that such an allocation always exists when $n = 2$. They also found an example with $n = 4$ in which no such allocation exists, but the example relies on measures that are not absolutely continuous with respect to each other. Brams, Jones, and Klamler [4] extended the problem to the case of proportional divisions among players with unequal entitlements, and showed that in this extended problem, envy-free, undominated allocations might fail to exist. But for $n \geq 3$, absolutely continuous measures, and equal entitlements, Gale's question has remained open.

In this paper we answer Gale's question in the negative when $n = 3$. In Section 2 we present a set of three measures on a pie that are absolutely continuous with respect to each other, and we show (in Section 3) that no allocation among players with these measures can be both envy free and undominated. The lesson of the pie-cutting model, therefore, is exactly opposite to the lesson of the cakes: envy-free, undominated divisions are not always possible.

Sections 4 and 5 provide motivation by showing how the example came about. Section 6 asks a new question about pies.

2 The example

In this section and the next we represent the pie as the interval $[0, 18]$ with the endpoints identified. Points in the interval are defined *mod 18*, so that, for example, $[15, 3]$ and $[15, 21]$ both represent the same connected interval. We call the players A, B, C, and their measures v_A, v_B, v_C (or generically, just v_i). Their respective pieces are always called S_A, S_B, S_C.

For describing the measures in the example we partition the pie into 18 sectors $[0, 1]$, $[1, 2]$,...,$[17, 18]$. We make each measure uniform over each sector, so that we can describe the measures by listing the values $v_i(k, k+1)$ for each player and each sector. With these conventions in place, here are the measures that form the example:

	0			3			6			9			12			15		18
v_A	130	100	69	131	70	100	100	70	131	69	100	130	60	100	100	100	100	60
v_B	100	100	115	5	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	115	5	100	100	100	100	100	100	100	100

Note that most sectors have value 100 for most players. We call the differences from 100, whether positive or negative, *excess values.* The following table shows the excess values. It is the same as the previous table, but with 100 subtracted from each cell.

	0			3			6			9			12			15		18
$v_A - 100$	+30		−31	+31	−30			−30	+31	−31		+30	−40					−40
$v_B - 100$			+15	−95														
$v_C - 100$									+15	−95								

One way to find a player's value of a piece is to multiply the piece's length by 100 and make adjustments based on the excess-values table.

3 Proof of the example

In this section we prove that these measures do not permit any envy-free, undominated allocation.

Given an allocation S_A, S_B, S_C, its *value vector* is the vector $(v_A(S_A), v_B(S_B), v_C(S_C))$ and its *total value* is $v_A(S_A) + v_B(S_B) + v_C(S_C)$; that is, the sum of the values of the pieces to the players that receive them.

In outline, the proof is as follows. First, any envy-free, undominated allocation must have total value at least 1800. Next, we show that there are only a few ways to achieve this sum: S_A, S_B, S_C can be

- Plan 1: $[0, 6]$, $[6, 12]$, $[12, 18]$;
- Plan 2: $[6, 12]$, $[12, 18]$, $[0, 6]$;
- Plan 3: $[3, 9]$, $[9, 15]$, $[15, 3]$;

or a slight modification of one of these plans.

Finally, we show that in Plan 1, some instance of envy occurs unless the pieces are exactly $[0, 6]$, $[6, 12]$, and $[12, 18]$, in which case the allocation is dominated by Plan 3. Plan 2 is similar. In Plan 3 as given, B envies C and C envies A, so that Plan 3 fails to be envy free. If Plan 3 is modified in such a way as to eliminate these instances of envy, its sum falls below 1800 and it is dominated by some variant of Plan 1 or Plan 2.

We now give the proof in greater detail.

1. In any envy-free allocation, each piece has length greater 5.5.

Proof: Each player's valuation of the entire pie is $v_i(0, 18) = 1720$. It follows that each player must consider its own piece to be worth more than 573, or it would necessarily prefer another player's piece to its own. By inspection, no piece of length less than or equal to 5.5 has value greater than 573 to any player. □

2. In any envy-free, undominated allocation, the total value is at least 1800.

Proof: Let a, b, c satisfy $a, b, c > 573$ and $a + b + c = 1800$. Consider the allocation

$T_A = [0, a/100]$;

$T_B = [a/100, (a + b)/100]$;

$T_C = [(a + b)/100, 18]$.

It is easy to check that for this allocation, the value vector $(v_A(T_A), v_B(T_B), v_C(T_C))$ is exactly equal to (a, b, c). Any envy-free allocation $\{S_i\}$ whose total value is less than 1800 is dominated by some allocation $\{T_i\}$ of this form. □

3. In any envy-free, undominated allocation, the values of the pieces must average 100 per sector, as valued by the players that receive them. □

This means that sum of the "excess values" assigned by the players to their own pieces must be at least 0. By inspection, B and C cannot contribute more than +15 each to this sum, and A cannot contribute more than +2.

4. In any envy-free, undominated allocation, C's piece cannot include any part of the interval $[7.5, 9]$.

Proof: For C and A to have pieces of suitable length, either C would need to accept all of $[9, 10]$ or A would need to accept all of either $[12, 13]$ or $[17, 18]$. The total value could not be kept from falling below 1800. □

5. In any envy-free, undominated allocation, B's piece cannot include any part of the sector $[2, 3]$.

Proof: B would need to avoid most of $[3, 4]$. Since C cannot overlap the interval $[7.5, 9]$, the piece to the right of S_B must be assigned to A. Now either A or C must accept the sector $[9, 10]$ (or they must share it), and this forces the total value below 1800. □

6. In any envy-free, undominated allocation, S_A must be of the form $[0, 6 + x]$ with $-1 \leq x \leq +1$, or $[6+x, 12]$ with $-1 \leq x \leq +1$, or $[3+x, 9+y]$ with $|x|+|y| \leq 2/31$.

Proof: We know from the last two steps that neither B nor C can contribute a positive excess value. Therefore A must contribute excess value of at least zero. By inspection, the only pieces of length greater than 5.5 that meet this standard are those identified. (We exclude pieces S_A with length 7 or more, since S_B or S_C would be too small.) □

The possible pieces for A define the three possible plans for envy-free, undominated allocations. We have shown that these are the only allocations in which each piece has length at least 5.5 and the pieces' total value is at least 1800. In each plan, we list S_A, S_B, S_C in order.

- Plan 1. $[0, 6 + x]$, $[6 + x, 12 + y]$, $[12 + y, 18]$ with $-1 < x, y < +1$;
- Plan 2. $[6 + x, 12]$, $[12, y]$, $[y, 6 + x]$ with $-1 < x, y < +1$;
- Plan 3. $[3+x, 9+y]$, $[9+y, 15+z]$, $[15+z, 3+x]$ with $|x|+|y| \leq 2/31$ and $-1 < z < +1$.

7. There is no envy-free, undominated allocation consistent with Plan 1.
Proof: We explore the consequences of envy-freeness:

- $v_A(S_A) \geq v_A(S_B)$ implies
$$200x - Ty \geq 0 \tag{1}$$
where T is either 60 or 130 depending on the sign of x.

- $v_B(S_B) \geq v_B(S_C)$ implies
$$-100x + 200y \geq 0. \tag{2}$$

- $v_C(S_C) \geq v_C(S_A)$ implies
$$-100x - 100y \geq 0. \tag{3}$$

Adding twice (3) to (1) forces $y \leq 0$ and adding (2) to twice (3) gives $x \leq 0$. Now (3) forces $x = y = 0$.

Therefore any envy-free variant of Plan 1 must be exactly $[0, 6]$, $[6, 12]$, $[12, 18]$, with value vector exactly $(600, 600, 600)$. But this allocation is dominated by $[3, 9]$, $[9, 15]$, $[15, 3]$ which delivers $(602, 600, 600)$. □

8. There is no envy-free, undominated allocation consistent with Plan 2.
This is symmetrical to Plan 1. □

9. There is no envy-free, undominated allocation consistent with Plan 3.
Proof: The problem here is that, with no adjustments to the boundaries, B envies C and C envies A. These conditions cannot both be fixed without reducing A's piece, but any sufficient reduction of A's piece causes the total value to fall below 1800.

More precisely: Suppose that the left end of C's piece is at least 15 (that is, $z \geq 0$). Then C's piece is at most $[15, 3 + x]$ where $x \leq 2/31$. Thus $v_C(S_C)$ is at most $600 + 100x < 607$. But $v_C(S_A)$ is at least $615 - 100x + 100y > 608$, so C envies A.

Suppose on the other hand that $z < 0$. Then $v_B(S_B)$ is at most $600 - 100y < 607$, while $v_B(S_C)$ is at least $615 + 115x > 607$. So B envies C. □

We have completed the proof that these measures do not allow an envy-free, undominated allocation.

4 The failed proof

This section and the next are aimed at explaining how the example came about.

As often happens, the example arose from exploiting the gap in a failed proof. In this section we outline a (false) proof that an envy-free, undominated allocation always exists. We consider only the case of $n = 3$.

For any allocation $\{S_A, S_B, S_C\}$ with value vector $(v_A(S_A), v_B(S_B), v_C(S_C))$ and total value $s = v_A(S_A) + v_B(S_B) + v_C(S_C)$, we define its *proportions vector* to be

$$(v_A(S_A)/s, v_B(S_B)/s, v_C(S_C)/s)\,.$$

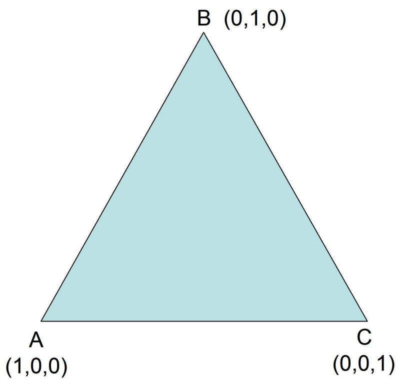


Figure 1: Simplex of proportion vectors

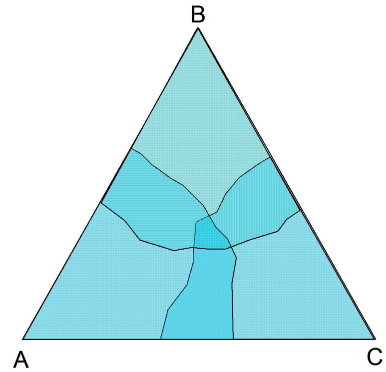


Figure 2: The sets K_A, K_B, K_C overlap

The possible proportions vectors form a simplex as shown in Figure 1. Formally the simplex is

$$K = \{(a, b, c) \quad | \quad a, b, c \geq 0 \text{ and } a + b + c = 1 \quad \}.$$

Each vertex represents an allocation of the whole cake to one player. These results are easily established:

1. Every vector $(a, b, c) \in K$ is the proportions vector of an undominated allocation.
2. In every undominated allocation, at least one player is satisfied (that is, not envious).

Now consider the sets

$$K_A = \{(a, b, c) \quad | \quad \text{There is an allocation with proportions vector } (a, b, c) \\ \text{in which player } A \text{ is satisfied } \}$$

and the corresponding sets K_B, K_C for players B and C. From the second result above, these sets cover S: that is, $K_A \cup K_B \cup K_C = S$. Further, K_A contains the vertex $(1, 0, 0)$ (because A is clearly satisfied with the whole cake) but does not intersect the side opposite $(1, 0, 0)$ (because in each of those allocations, A has an empty piece). Similar facts hold for K_B and K_C. If K_A, K_B, K_C are not closed sets, replace them with their closures.

Under these circumstances, a topological lemma like the one used in [10] applies. The lemma then says that K_A, K_B, K_C must have a point in common (Figure 2). Doesn't that point represent an undominated allocation in which all three players are satisfied? Doesn't that imply that there is an envy-free, undominated allocation?

So goes the proof.

The flaw is that this "proof" doesn't give the result we want. It actually proves (at best) that there is a proportions vector such that every player is satisfied by *some* undominated allocation having that proportions vector. But there might be two or more undominated allocations for the same proportions vector. Then A might be satisfied with one of the allocations, while B and C are satisfied with another one. There might then be no single undominated allocation that satisfies everybody.

This tells us one feature we should look for in an example: there should be at least one instance of two different undominated allocations with the same proportions vector, which satisfy different sets of players. In the next section we build an example around this feature.

5 Building an example

Strategy. Our strategy in building an example is to start with a case in which every envy-free allocation has the same proportions vector, $(1/3, 1/3, 1/3)$. In fact, in the case we choose there are two different envy-free allocations, both involving the same pieces but assigned to different players. Both allocations are "brittle" in the sense that small modifications keep them from being envy free. This situation is illustrated in Figure 3.

We then add an overlay of additional allocations as shown in Figure 4. The allocations in the overlay have the same proportions vectors as the old ones, and dominate them. In particular, the envy-free allocations that we started with are dominated by allocations in the overlay.

We need to prevent new envy-free allocations from arising on the overlay. We do this by making B envy C and C envy A for all allocations in the overlay. These conditions could be fixed by making A's piece smaller—that is, by shifting the proportions vector toward the lower left in the figure. To prevent this, we let the overlay sink below the former layer, so that any new envy-free allocations are dominated by allocations in the original example. Along the edge where the overlay intersects the original layer, there are two undominated allocations for each proportions vector.

Proceding in steps. We begin the construction with this simple example.

Version 1: Two brittle envy-free allocations—

	0			3			6			9			12			15		18
v_A	100	100	100	100	100	100	100	100	100	100	100	100	5	5	5	5	5	5
v_B	5	5	5	5	5	5	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	5	5	5	5	5	5	100	100	100	100	100	100

In this version, suppose that cuts are made at 0, 6, and 12. Then each player is indifferent between two pieces: A is happy with $[0, 6]$ or $[6, 12]$, B is happy with $[6, 12]$ or $[12, 18]$, and C is happy with $[12, 18]$ or $[0, 6]$. This gives us two envy-free, undominated solutions:

- Plan 1: $[0, 6]$, $[6, 12]$, $[12, 18]$ to A, B, C respectively.
- Plan 2: $[6, 12]$, $[12, 18]$, $[0, 6]$ to A, B, C respectively.

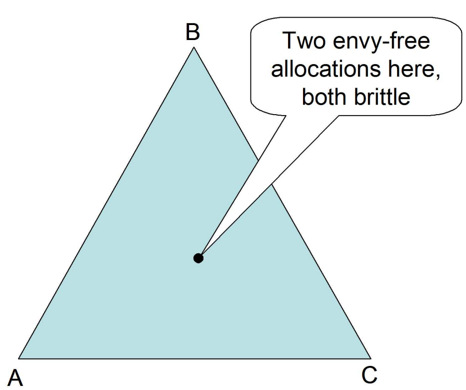


Figure 3: Starting the construction

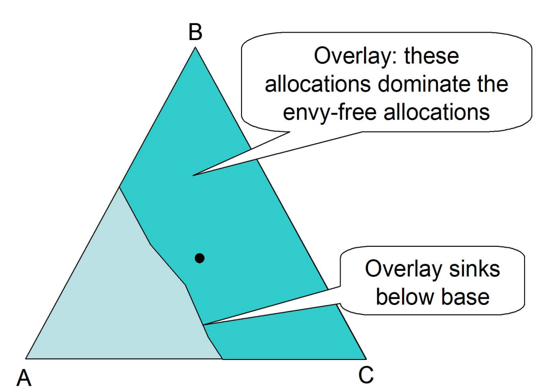


Figure 4: Overlay

Each of these allocations delivers values of 600, 600, 600 to the respective players. This is clearly the best that can be done. Since each of these allocations has three players at the very edge of envy, these allocations are "brittle" in the sense described above.

Unfortunately, Version 1 allows additional envy-free allocations, and they are not brittle. In Version 1, any allocation of the form $[x, 6+x]$, $[6+x, 12+x]$, $[12+x, x]$ with $0 \leq x \leq 6$ is envy-free and undominated. In order to kill off most of these allocations, we add some "decorations" to A's measure:

Version 2: Poisoning some of the extra envy-free allocations—

	0			3			6			9			12			15		18
v_A	$\underline{130}$	100	$\underline{70}$	100	100	100	100	100	100	$\underline{70}$	100	$\underline{130}$	5	5	5	5	5	5
v_B	5	5	5	5	5	5	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	5	5	5	5	5	5	100	100	100	100	100	100

This change causes most of the unwanted allocations to be undesirable, because A will not want to accept one of the "70" sectors without the nearby "130" sector. There are still a continuum of unwanted envy-free allocations, but most of them are now dominated by Plans 1 and 2.

Before moving on, we modify the "killer sectors" in all three measures. This change adds flexibility that we will need later.

Version 3: Adjusting the killer sectors—

	0			3			6			9			12			15		18
v_A	130	100	70	100	100	100	100	100	100	70	100	130	$\underline{60}$	100	100	100	100	$\underline{60}$
v_B	100	100	100	$\underline{5}$	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	100	$\underline{5}$	100	100	100	100	100	100	100	100

Note that in Versions 2 and 3 there is still one extra envy-free allocation that offers values of 600, 600, 600 to the players:

- Plan 3: $[3, 9]$, $[9, 15]$, $[15, 3]$ to A, B, C respectively.

We'll preserve this allocation, and use it to dominate the allocations from Plan 1 and Plan 2. To do this, we freeze it in place with another set of "decorations," and then shift 1 extra point of A's measure into each sectors $[3, 4]$ and $[8, 9]$ to make them more attractive. (Plan 3 and its variants form the "overlay" shown in Figure 4.)

Version 4: Beefing up the "Plan 3" allocation—

	0			3			6			9			12			15		18
v_A	130	100	$\underline{69}$	$\underline{131}$	$\underline{70}$	100	100	$\underline{70}$	$\underline{131}$	$\underline{69}$	100	130	60	100	100	100	100	60
v_B	100	100	100	20	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	100	20	100	100	100	100	100	100	100	100

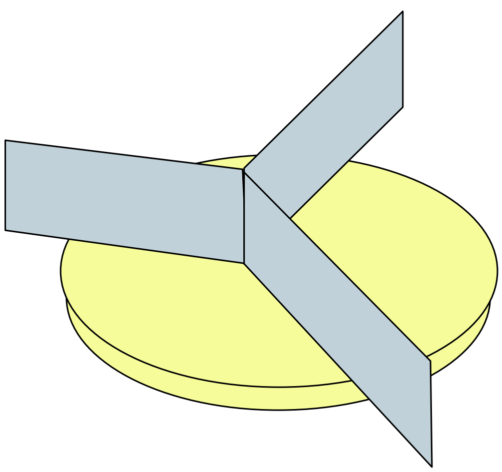
Now Plan 3 offers values of 602, 600, 600 to the respective players, so it dominates both Plan 1 and Plan 2. Note that any modification to Plan 3 that alters A's piece by more than a trivial amount will reduce its value to A below 600, which can keep it from being undominated.

As it stands, Plan 3 is both envy free and undominated. Finally, we modify it by making B envy C and C envy A.

Version 5, final: Making players envious in Plan 3—

	0			3			6			9			12			15		18
v_A	130	100	69	131	70	100	100	70	131	69	100	130	60	100	100	100	100	60
v_B	100	100	$\underline{115}$	5	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	$\underline{115}$	5	100	100	100	100	100	100	100	100

Of course these instances of envy could be corrected—but only by significant alterations of A's piece, which would cause Plan 3 to be dominated. So the example is now complete.



6 A fresh problem

A pie-cutter has three blades, fixed at angles of 120 degrees. We can place its center anywhere we like, and orient it as we please as long as the blades remain vertical and their angles remain fixed. Given three players with preferences defined by absolutely continuous measures, can we use the pie-cutter to find an envy-free allocation? Can we find an allocation that is both envy free and undominated?

The blades form a Steiner tree, which suggests a generalization to more then three players. Can a pie be divided fairly among n players, with piece boundaries that form a Steiner tree? That is, that meet each other only at 120 degree angles?

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Swarthmore College, Swarthmore, PA 19081
`mail@walterstromquist.com`