Packing Layered Posets Into Posets

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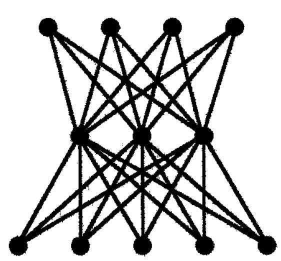
This essay is really about packing numbers of permutations, but the part that has been written so far is about packing posets. This work was inspired by Herb Wilf's questions about packing densities of permutations.

Definitions: Layered and LOT posets and lower sets.

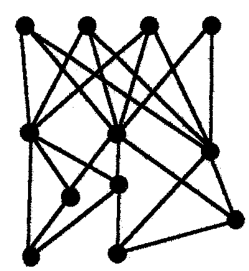
A **poset** is a nonempty, finite set with a partial ordering denoted as usual by $\geq$, $\leq$, $>$, $<$. We will use the terms "above" and "below" to refer to the relationship between elements in the poset ordering.

A poset P is **layered** if it can be partitioned into disjoint subsets $P_1,\ldots,P_n$ such that $x>y$ in P iff $x \in P_i$ and $y \in P_j$ with $i>j$. That is, elements in each layer are above all elements in lower layers, but distinct elements in the same layer are incomparable.

A poset is **layered on top (LOT)** if it can be partitioned into two disjoint subsets P_1 and P_2, with P_2 nonempty, such if x is in P_2, we have $x>y$ iff y is in P_1. Note that if this is the case, then P_2 must consist precisely of the maximal elements of P. Thus, the LOT condition means that every maximal element of P is above every non-maximal element. Every layered poset is also LOT.



Typical layered poset: Full comparability between layers, none within a layer.



Typical layered-on-top (LOT) poset: Every maximal element is above every other element.

If x is a maximal element of P, we denote by $L(x)$ the set $\{y \text{ in } P \mid y < x\}$. We call this the **lower set** corresponding to x. Note that a poset is LOT iff all maximal elements have the same lower set, and that in this case the lower set is just the set of non-maximal elements.

Packing numbers and packing functions. When posets are "optimal."

Let P and Q be posets. Then the **packing number** of P in Q, denoted **p(P,Q)**, is defined to be the number of distinct subsets of Q (with ordering inherited from Q) that are isomorphic to P. Viewed as a function of Q, we call $p(P,Q)$ a **simple packing function**, or the **simple packing function corresponding to** ***P***, and P is called its **pattern.** More generally, a **packing function** is any non-negative linear combination of simple packing functions:

$$f(Q) = a_1 p(P_1,Q) + \cdots + a_m p(P_m,Q)$$

where m is a non-negative integer, $P_1,\ldots,P_m$ are posets, and $a_1, \ldots, a_m$ are non-negative real numbers. The a_i's are called the **coefficients** of this packing function, and the P_i's are called the **patterns.**

In this paper we will be looking for posets Q that maximize a given packing function, given the size of Q. We say that Q is **optimal for f** if f(Q) >= f(Q') for any poset Q' satisfying |Q'| = |Q|. If |Q| = n, we also say that Q is **optimal for f of size n.**

We are mainly concerned with simple packing functions. It turns out that our main results are true for general packing functions, and that proving them in that form makes the induction steps easier.

Packing functions with LOT patterns are optimized by LOT posets.

Theorem 1. Let $n \geq 1$, let P be a LOT poset, and let f be the simple packing function defined by $f(Q) = p(P,Q)$. Then there is a LOT poset Q which is optimal for f of size n.

Proof. We need some specialized notation. If P and Q are posets with x, y in Q, then we write p(P,Q,x) for the number of subsets of Q that contain x and are isomorphic to P, and we write p(P,Q,not y) for the number of subsets of Q that are isomorphic to P but do not contain y. We combine these notations freely and extend them to general packing functions in the obvious way.

Now let f(Q)=p(P,Q) as in the statement of the theorem. Let us choose a specific poset Q which is optimal for f of size n. Specifically, we choose Q so that among all of the posets that are optimal for f of size n, Q has the largest possible group of maximal elements with identical lower sets. We will suppose that Q is not LOT, and argue to a contradiction.

If Q is not LOT, then we can choose maximal elements x and y of Q with L(x) ≠ L(y). We choose x and y so that exactly one of them is in that large group of maximal elements with identical lower sets. We will construct another poset Q' with the same underlying set as Q, but which contradicts the choice of Q.

Without loss of generality, assume that

$$f(Q, x, \text{not } y) >= f(Q, y, \text{not } x). \qquad [1]$$

If equality holds in [1], then we can also insist that x is the one which is in the large group with identical lower sets.

Now we construct Q' as follows: If u and v in Q are both different from y, then v>u in Q' iff v>u in Q. If u is different from y, then y>u in Q' iff x>u in Q. Thus, we have altered the relationships involving y in order to make L(y) agree with L(x) in Q', and left the relationships not including y alone.

Now:

$$f(Q') = f(Q', \text{not } y) + f(Q', y, \text{not } x) + f(Q', x, y).$$

Let's work on each of these terms separately. For the first term, clearly

$$f(Q', \text{not } y) = f(Q, \text{not } y),$$

since for subsets not including y, the posets Q and Q' are identical. Furthermore, from the construction of Q', we have

$$f(Q', y, \text{not } x) = f(Q, x, \text{not } y) >= f(Q, y, \text{not } x).$$

(The first equality comes from counting subsets isomorphic to P: for each such subset of Q' that contains y but not x, we can put x in place of y, and get an isomorphic subset of Q.)

Finally, we derive

$$f(Q', x, y) >= f(Q, x, y)$$

from the layered character of P, as follows: Suppose A is a subset of Q' that is isomorphic to P and contains x and y. Then the elements of P corresponding to x and y are both maximal elements of P. All other maximal elements of P must be mapped to
Q' - {x,y} - L(x) - L(y), and all non-maximal elements of P must be mapped to L(x) $\cap$ L(y).
But that means that the entire image of P---that is, all of A---is in the part of Q' whose ordering is unchanged from Q. So, A is still isomorphic to P when A is viewed as a subset of Q.

We conclude that

$$\begin{aligned} f(Q') &= f(Q', \text{not } y) + f(Q', y, \text{not } x) + f(Q', x, y) \\ &\geq f(Q, \text{not } y) + f(Q, y, \text{not } x) + f(Q, x, y) = f(Q). \end{aligned} \qquad [2]$$

Furthermore, if we have a strict inequality in [1], then we have a strict inequality in [2] as well. But that contradicts the optimality of Q. On the other hand, if we have equality in [1] and [2], then Q' and Q are both optimal, but in this case Q' has a larger group of maximal elements with identical lower sets (since y has been brought into the group).

This completes the proof of Theorem 1. //

(Note that the above construction might convert non-maximal elements of Q into maximal elements of Q'. This does not affect the proof.)

Theorem 2 is the same as theorem 1, but for general packing functions.

Theorem 2. Let $P_1, \ldots, P_m$ all be LOT posets, and consider the packing function

$$f(Q) = a_1\, p(P_1,Q) + \ldots + a_m\, p(P_m,Q).$$

Let $n \geq 1$. Then there is a LOT poset Q which is optimal for f of size n.

Proof: The proof is identical to the proof of Theorem 1. //

(Note that theorem 2 does not follow directly from the statement of theorem 1.)

Packing functions with layered patterns are optimized by layered posets.

Theorems 3 and 4 are the same as theorems 1 and 2, but with "layered" in place of "LOT". We will give the proof of Theorem 3. The proof of theorem 4 is an obvious extension of the same proof.

Theorem 3. Let $n \geq 1$, let P be a layered poset, and let f be the simple packing function defined by $f(Q) = p(P,Q)$. Then there is a layered poset Q which is optimal for f of size n.

Theorem 4. Let $P_1, \ldots, P_m$ all be layered posets, and consider the packing function

$$f(Q) = a_1\, p(P_1,Q) + \ldots + a_m\, p(P_m,Q).$$

Let $n \geq 1$. Then there is a layered poset Q which is optimal for f of size n.

Proof of Theorem 3: Induct on the size n.

Let Q be optimal for f of size n. From theorem 1, Q must be layered on top. Thus Q consists of some maximal elements which are above all non-maximal elements, and a set Q_0 of non-maximal elements. If Q_0 is empty, then Q is actually layered and we are done; so we may assume that Q_0 is non-empty; and in fact Q_0 is a poset of size smaller than n.

If we alter the ordering of Q_0 in such a way that it remains a poset, then Q also remains a poset.

Let u1, u2,... be the maximal elements of P, and let x1, x2,... be the maximal elements of Q. Then u1, u2, ... are all interchangeable, and x1, x2,... are, too.

For each value of $i \geq 0$, consider the number of subsets of Q which are isomorphic to P and which contain exactly i maximal elements of Q. In each of the isomorphisms, exactly i of the maximal elements of P correspond to the maximal elements of Q---it doesn't matter which maximal elements in either case---and the rest of P corresponds to elements of Q_0. Thus, the number of such packings is

(number of maximal elements of Q)-choose-(i), times

the number of packings of (P with i of its maximal elements removed) in Q_0.

It follows that p(P,Q) is just the sum of these terms for i >= 0.

But this sum is just a (general) packing function with Q_0 as its argument. (Actually, one of the terms in the sum is a constant independent of Q_0, but that doesn't matter.) All of the subsets of P involved in the packing function are layered. If the ordering on Q is such as to maximize p(P,Q), it must be that the ordering on Q_0 is such as to maximize this packing function. By the induction assumption, Q_0 must be layered. So, Q itself is layered. This completes the proof of Theorem 3. //

Example, and the connection to permutations.

Let Q be a sequence of n distinct numbers $q_1,\ldots,q_n$. We might as well suppose that Q is a permutation of 1..n. A "132 pattern" in Q is a triple (i,j,k) with $i < j < k$ but $q_i<q_k<q_j$. How can Q be chosen so as to maximize the number of its 132 patterns?

This is the simplest case of a question asked by Herb Wilf. Actually the answer to this special case has been known for a long time. But we can generalize by regarding "132" as itself a permutation, and asking the same question with any other permutation π, say of size k, in its place.

There is an obvious upper bound n-choose-k, independent of the pattern π, but it cannot be realized for most patterns π. Wilf defines the packing density of the pattern π as

limit as n goes to infinity of

maximum over all permutations Q of size n of the number of π-patterns in the permutation Q, divided by n-choose-k,

if it exists. The first open question is whether this limit always exists. (Surely it must! But the proof has not appeared.) The next questions are to determine the density for particular patterns π, and to find out which permutations Q achieve the maxima in the definition.

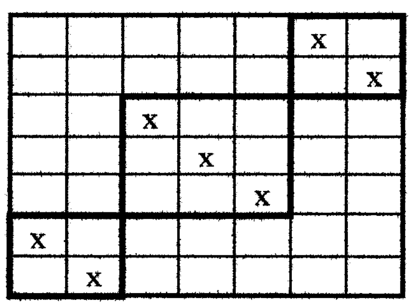
We will show how Theorems 3 and 4 can be used to compute the packing densities of certain "layered" permutations π.

For any permutation π of 1..k, we can construct a poset P as follows: Its elements are the pairs (i, $\pi(i)$), with the usual two-dimensional ordering. We say that π is **layered** (or **LOT**) if P is layered (or LOT). In general the correspondence is not 1-1: One poset might correspond to more that one permutation, and many posets (those of dimension greater than 2) do not correspond to any permutation. But for layered permutations and posets, the correspondence is a bijection.

A layered permutation consists of one or more blocks, with the elements in each block higher than the elements of all previous blocks, but decreasing within the block. For example,

3 2 1   5 4   9 8 7 6   or   6 5 4 3 2 1   7 .

The identity permutation is layered. So is the permutation 132. The graph of a layered



permutation looks like this:

For layered permutations, the π-patterns in Q correspond exactly to the P-patterns in the poset corresponding to Q. We have seen that for layered posets, packing densities are maximized when Q is layered. The same is true for permutations: packing densities are maximized by layered permutations Q.

In the next section, we use this fact to compute the packing density of 132, which turns out to be

$$2\sqrt{3} - 3 \approx 0.464.$$

The Packing Density of the Permutation 132

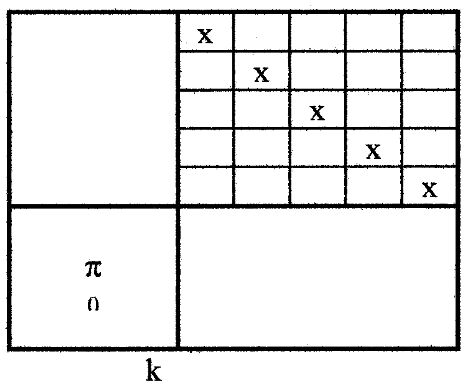
Let $P(132,\pi)$ denote the number of 132 patterns in a permutation π, and let $f(n)$ be defined by

$$f(n) = \max_{|\pi| = n} P(132, \pi).$$

Then the packing density of 132 is the limit of f(n)/(n-choose-3) as n goes to infinity.

Fix n. We will find a permutation π of size n that maximizes $P(132,\pi)$, and thus determines the value of f(n).

Since 132 is layered, the above theorems tell us that we need only consider layered permutations π. Thus π has a graph of the following form:



The figure shows π as consisting of a top layer of size (n-k), and lower layers that form a permutation π_0 of size k. Inspection shows that the number of 132 patterns in π is

(a) the number of 132 patterns in π_0, plus

(b) the number of triples consisting of two elements from the top layer and one from any lower layer.

It is clear that the choice of π_0 affects only (a), and so we should choose π_0 to optimize the number of 132 patterns for size k.

We have, therefore, a recursion for f(n):

$$f(n) = \max_{k < n} \left[f(k) + k \binom{n-k}{2} \right].$$

The solution to this recursion involves two numbers that we will call a and b:

$$a = \frac{1}{2}\left(\sqrt{3} - 1\right) \approx 366, \quad \text{and}$$

$$b = 2\sqrt{3} - 3 \approx 464.$$

Our idea is to choose k equal to about an, with the result that f(n) is about $b(n^3/6)$. The packing density is just b.

Lemma: If f(n) solves the above recursion, then $f(n) \leq b(n^3/6)$ for all $n \geq 1$.

Proof: By induction on n.//

Lemma: If f(n) solves the above recursion, then $f(n) \geq b(n^3/6) - 5n^2$ for all $n \geq 1$.

Proof: By induction on n, choosing k to be the greatest integer not exceeding an.//

It follows that the limit of $f(n)/(n^3/6)$, which is the same as the limit of f(n)/(n-choose-3), which is the same as the packing density of 132, is just b.

Packing Densities of Other Permutations

The packing densities of the permutations 123 and 321 are 1. Symmetry arguments show that the packing densities of 213, 231, and 312 are all the same as that of 132, namely b.

Of the 24 permutations of order 4, eight are not layered and we have nothing to say about them. The others fall into symmetry classes, each of which contains at least one layered permutation. These groups are represented by 1234 (whose packing density is 1) and 1432, 1243, 1324, and 2143.

The case of 1432 can be handled exactly like 132. We write:

a = unique positive solution to $(x^3 + x^2 + x) = \frac{1}{3}$; thus, $a \approx 2531$;

$$b = \frac{4a(1-a)^3}{(1-a^4)} \approx 42357.$$

Then the packing density of 1432 is just b, realized by permutations of the type illustrated above with k about equal to an.

We conjecture that this pattern continues: The packing density of the permutation

1 M (M-1) ... 2

is calculated by

a = unique positive solution to (x^(M-1)+...+x)=1/M;

b = M a (1-a)^(M-1) / (1-a^M).

This requires checking. In fact, this being a draft, everything in it requires checking.

The case of 2143 does not yield to the same recursion technique. But we still need to look only at layered permutations, and it is easy enough to verify that the optimal permutations for packing 2143 are symmetrical two-block permutations like

4321 8765.

The packing density turns out to be 3/8.

The case of 1243 illustrates the difficulties we still have ahead of us. In fact, I can't figure out the packing densities of 1243 or 1324, or even prove that they exist. The argument starts easily enough, following the argument for 132, but when it comes to choose π_0, the simple recursion fails. In fact π_0 is not the permutation that maximizes the packings of 1243 for size k; instead, it maximizes some linear combination of simple packing functions, and the linear combination changes with each n.

A reasonable conjecture is that 1243 is best packed in two-block permutations like

1234 8765

(at least for large n) and that its packing density is 3/8. The case of 1324 is much trickier and I have no conjecture.

Summary: We have succeeded in computing packing densities for some permutations. We have shown that for some permuations (the layered ones), we need only consider packing them into permutations of the same special form. And, we have found a way to look at the problem in the context of partially ordered sets.

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