Joint Packing Densities and the Great Limit Shape

(layered version)

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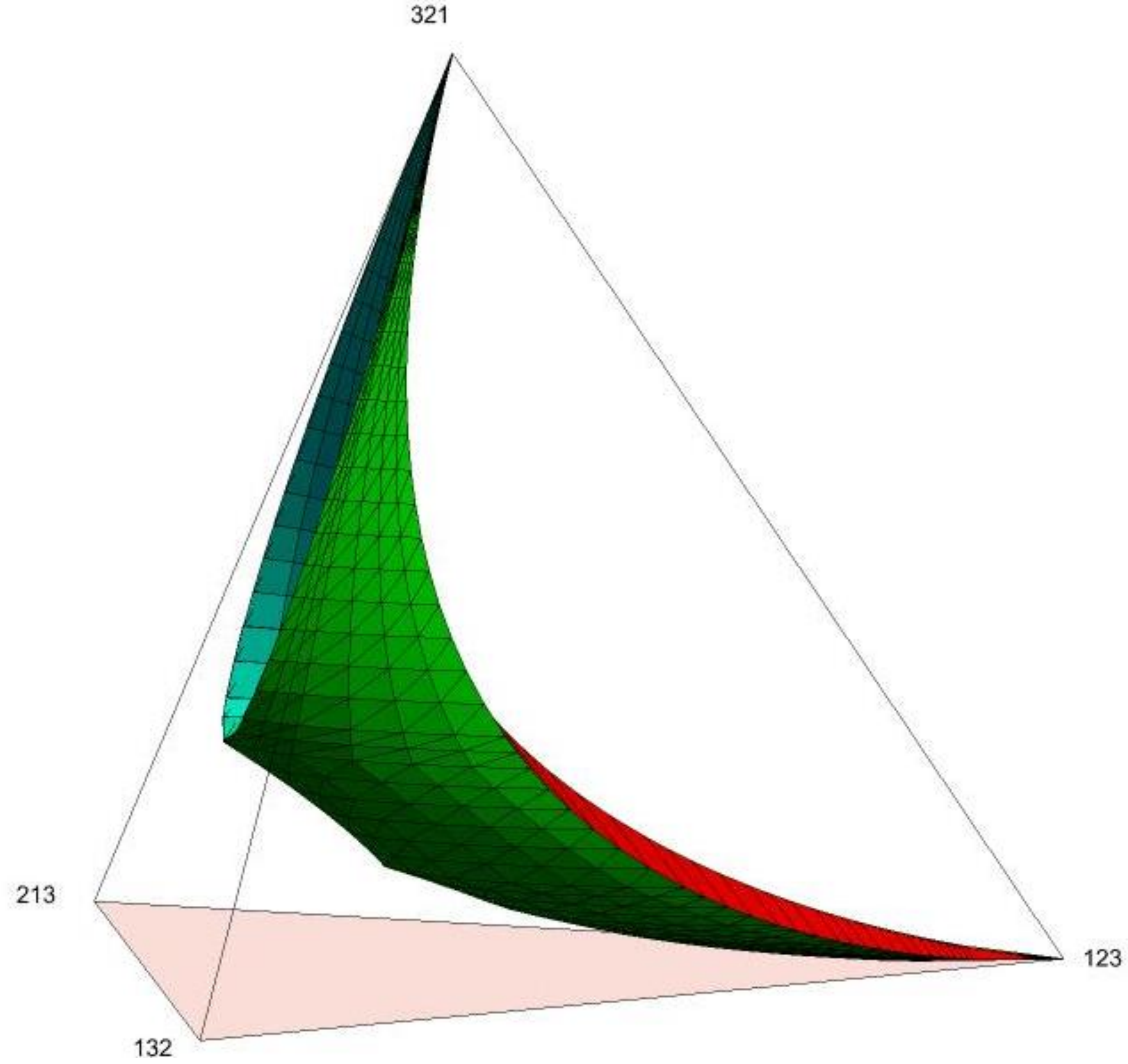
Great Limit Shape (layered version)



Outline

What does the shape represent?

1D -- 2D -- 6D -- 4D -- 3D

What shape is it?

Side view

Cross sections -- Front, Back, Below

Sides

Mysterious Gaps

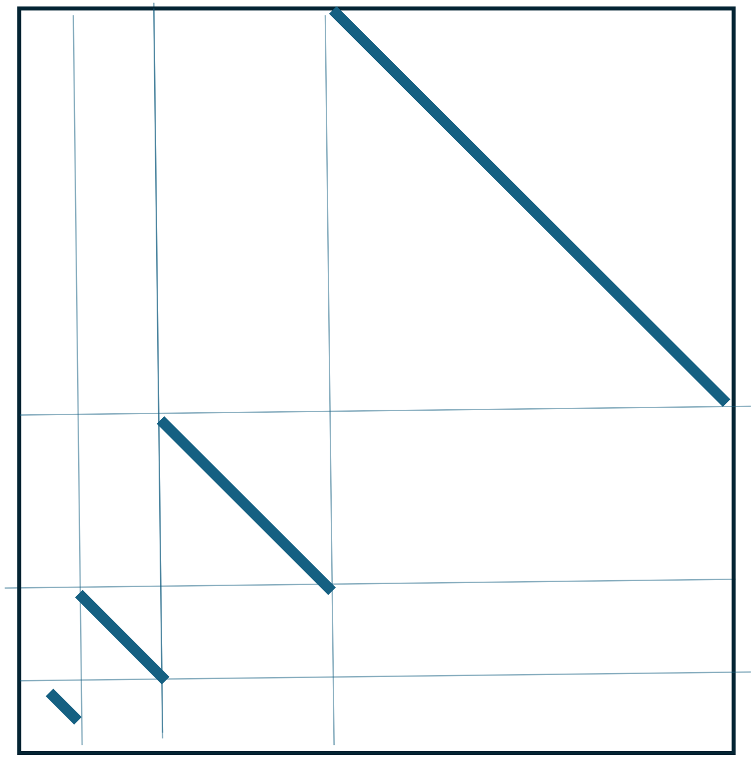
Packing Density

If $\sigma \in S_k$ (the "pattern") and $\pi \in S_n$, then the *packing density of* σ *in* π is

$$\delta_\sigma(\pi) = \frac{\text{the number of occurences of } \sigma \text{ in } \pi}{\binom{n}{k}}.$$

What values can occur? We care about the limit as $|\pi| \to \infty$.

Packing Density -- Example



$$\delta_{132}(\pi) = 2\sqrt{3} - 3 \approx 0.464\ldots$$

Joint Packing Densities

The *joint packing density* of two patterns, say 123 and 321, is a pair of numbers:

$$(\ \delta_{123}(\pi),\quad \delta_{321}(\pi)\ ).$$

What values can occur for this pair?

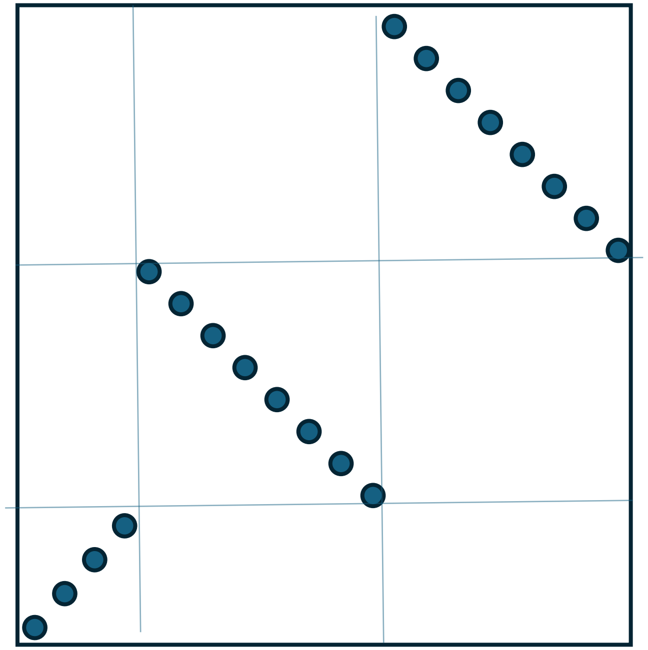
We know that (in the limit as $|\pi| \to \infty$) these densities satisfy

$$\delta_{123} + \delta_{321} \geq {}^{1}/_{4}\,.$$

Either coordinate can be zero, but the coordinates can't both be close to 0.

Joint Packing Densities

Example: $\pi \in S_{20}$



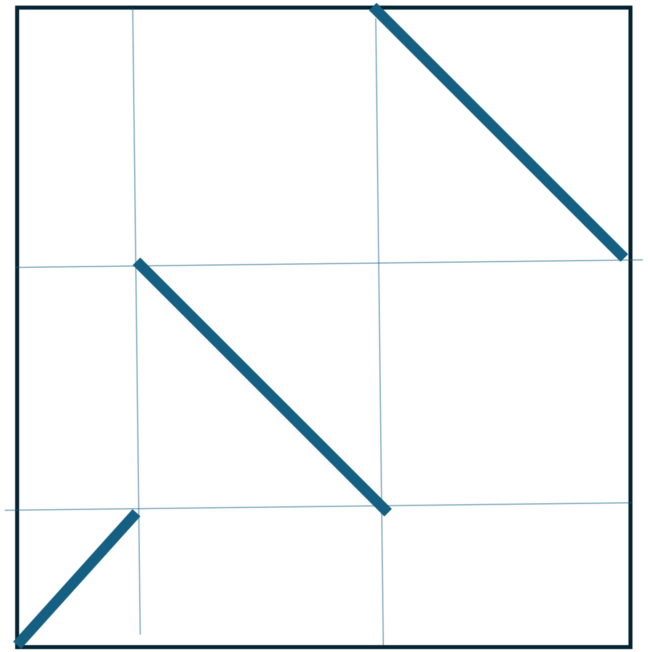
$$\delta_{123} = \frac{356}{1140} \approx 0.312$$

$$\delta_{321} = \frac{112}{1140} \approx 0.098$$

$(\delta_{123}(\pi), \ \delta_{321}(\pi)) = (0.312, 0.098)$.

Joint Packing Densities

Example:



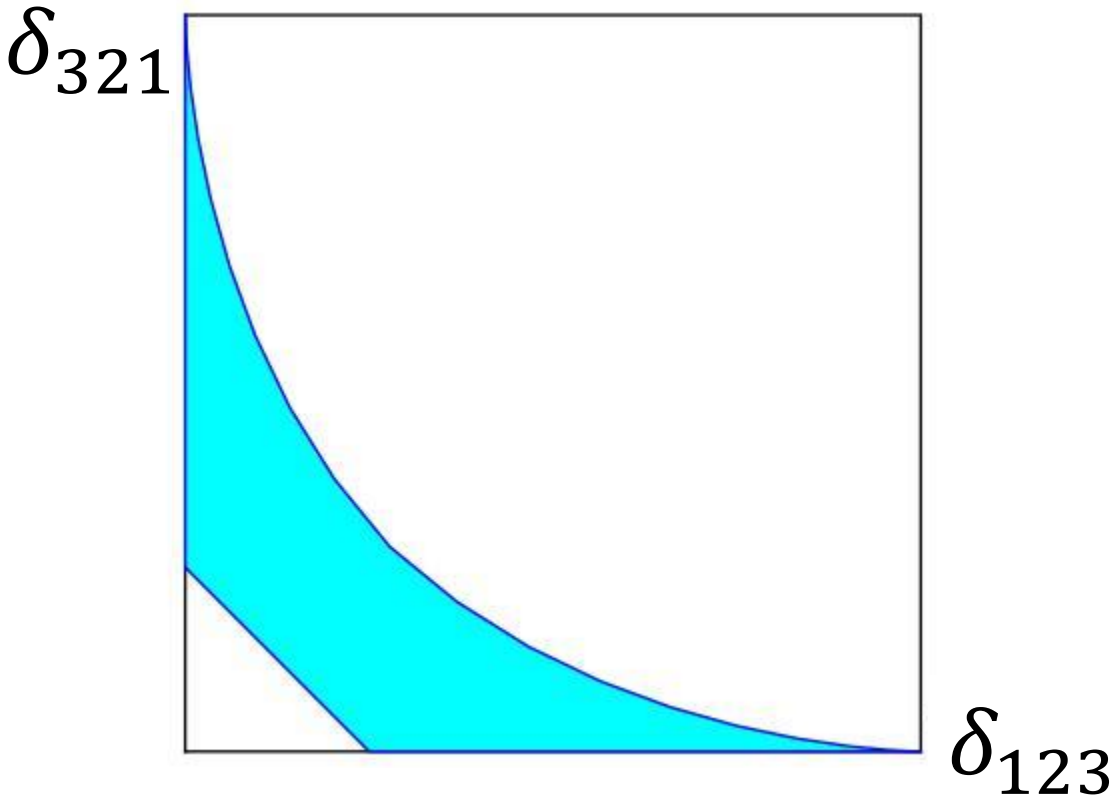
$$\delta_{123} = \frac{37}{125}$$

$$\delta_{321} = \frac{16}{125}$$

$(\delta_{123}, \delta_{321}) = (0.296, 0.128)$.

WHICH PAIRS CAN
ARISE IN THIS WAY?

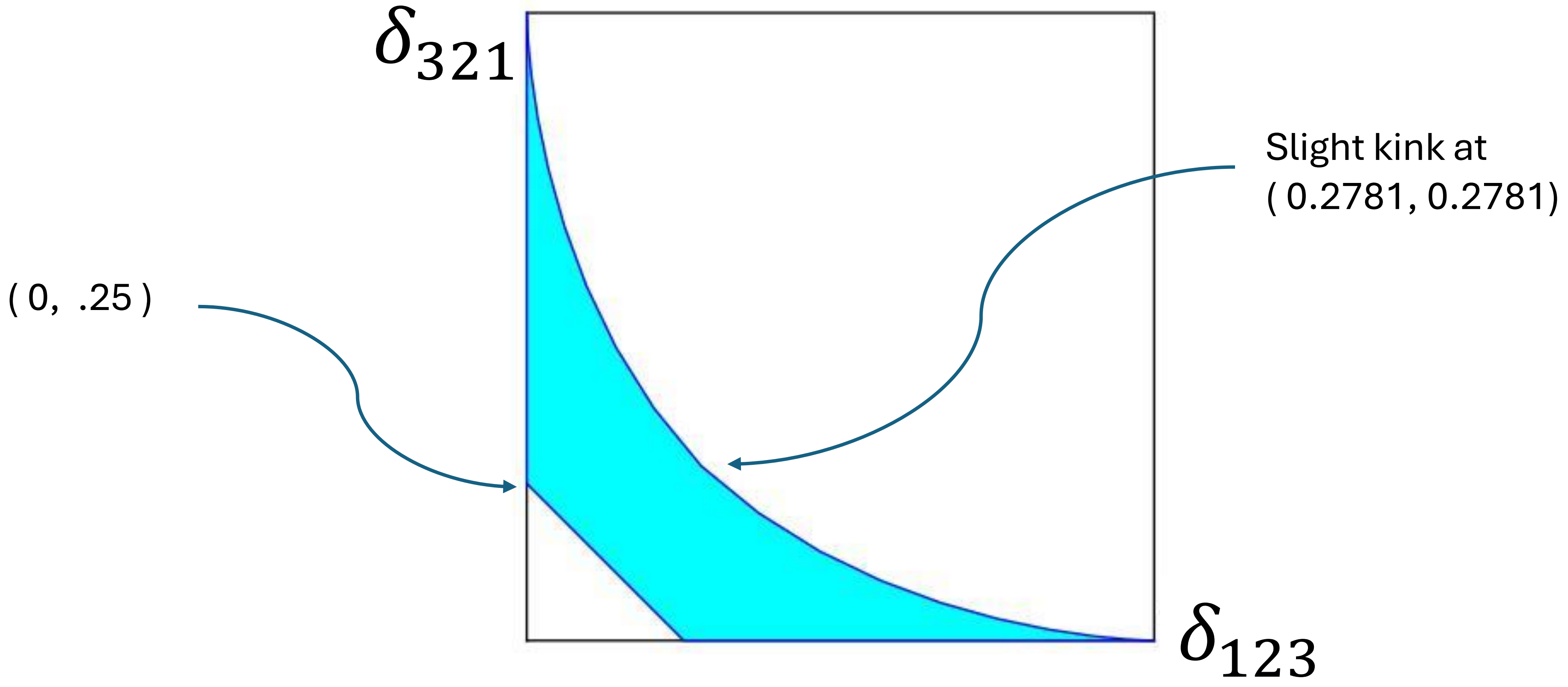
Joint Packing Density for δ_{123} and δ_{321}



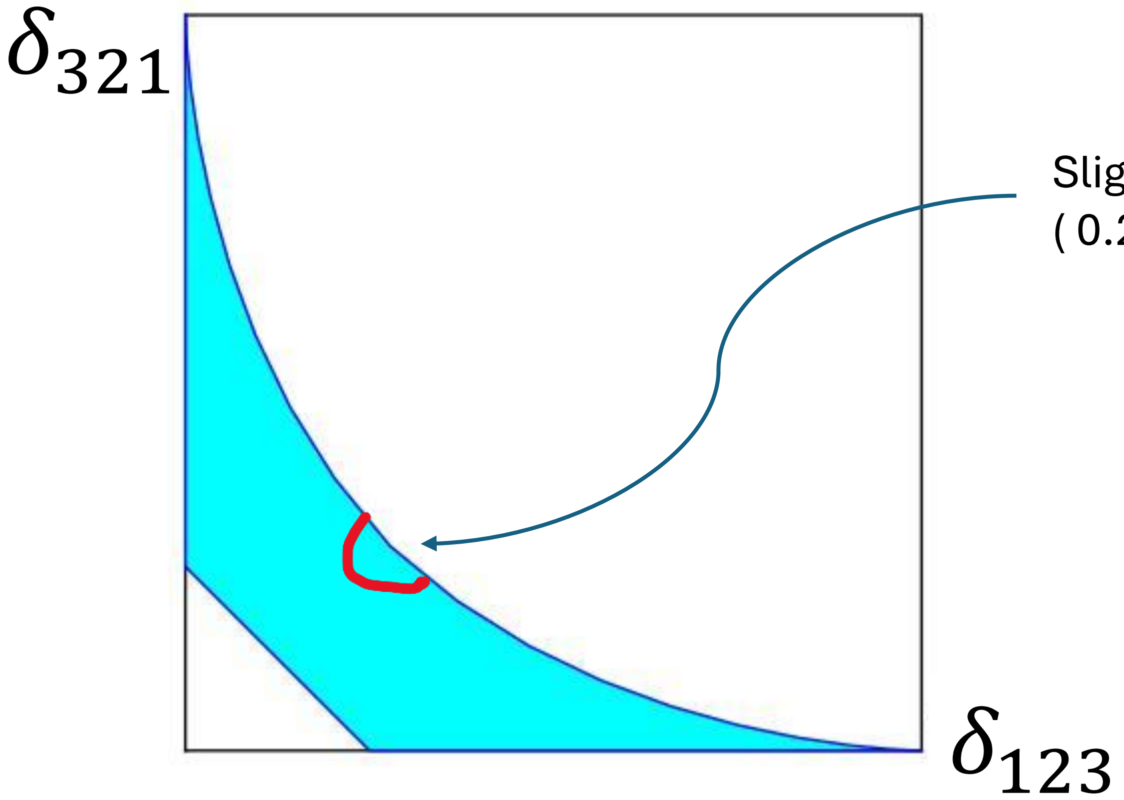
Possible limiting values of the pair (σ_{123}, σ_{321})

This diagram and proof of its correctness are due to Sergi Elizalde and his collaborators.

Joint Packing Density for δ_{123} and δ_321



Joint Packing Density for δ_{123} and δ_321



Slight kink at (0.2781, 0.2781)

This path in R2 is not the image of any continuous path among permutons.

The Great Limit Shape

Which vectors

$$v = (\delta_{123}, \delta_{132}, \delta_{213}, \delta_{231}, \delta_{312}, \delta_{321}) \in R^6$$

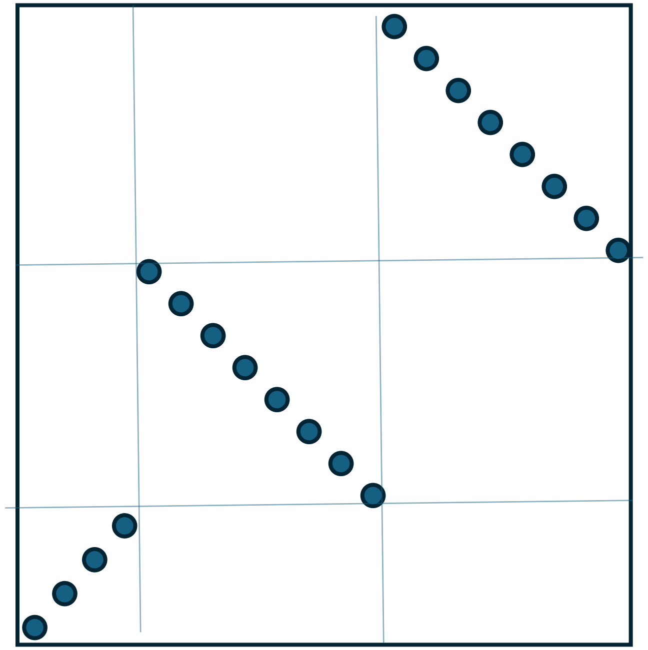
can occur as a limit of vectors

$$(\delta_{123}(\pi_i), \delta_{132}(\pi_i), \delta_{213}(\pi_i), \delta_{231}(\pi_i), \delta_{312}(\pi_i), \delta_{321}(\pi_i))$$

for a sequence of permutations $\pi_1, \pi_2, \pi_3, \ldots$ of increasing size?

Packing Vector

Example: $\pi \in S_{20}$.

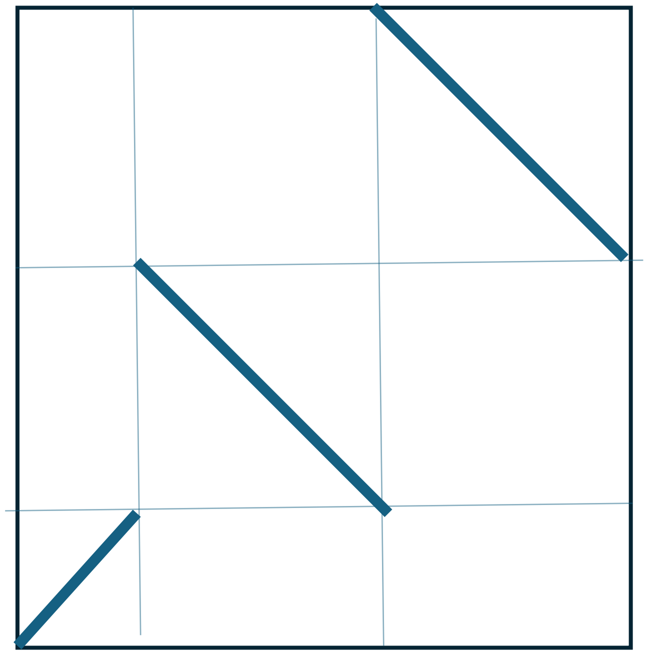


Pattern	123	132	213	231	312	321	
Number of occurrences	356	448	224	0	0	112	total 1140
Fraction of occurrences	.312	.393	.197	0	0	.098	

Packing Vector: (.312, .393, .197, 0, 0, .098)

Packing Vector --- Permuton

Example:

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The answer is a compact subset of R^6.

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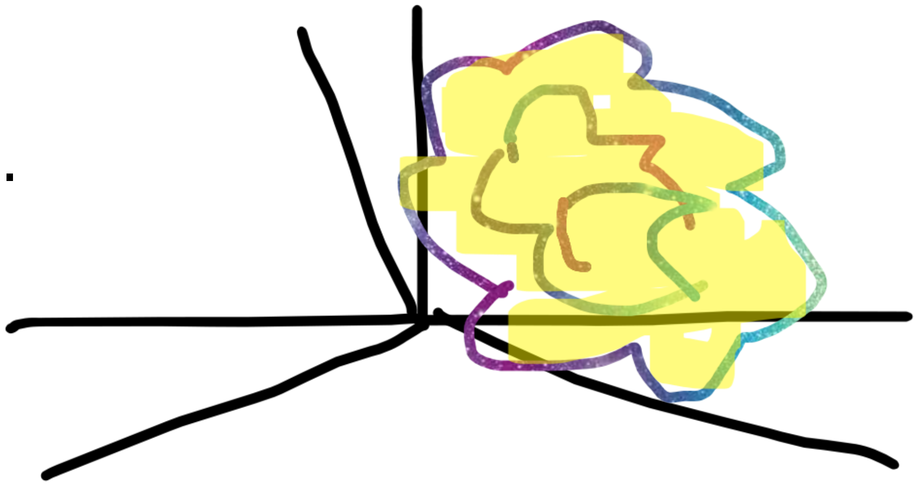
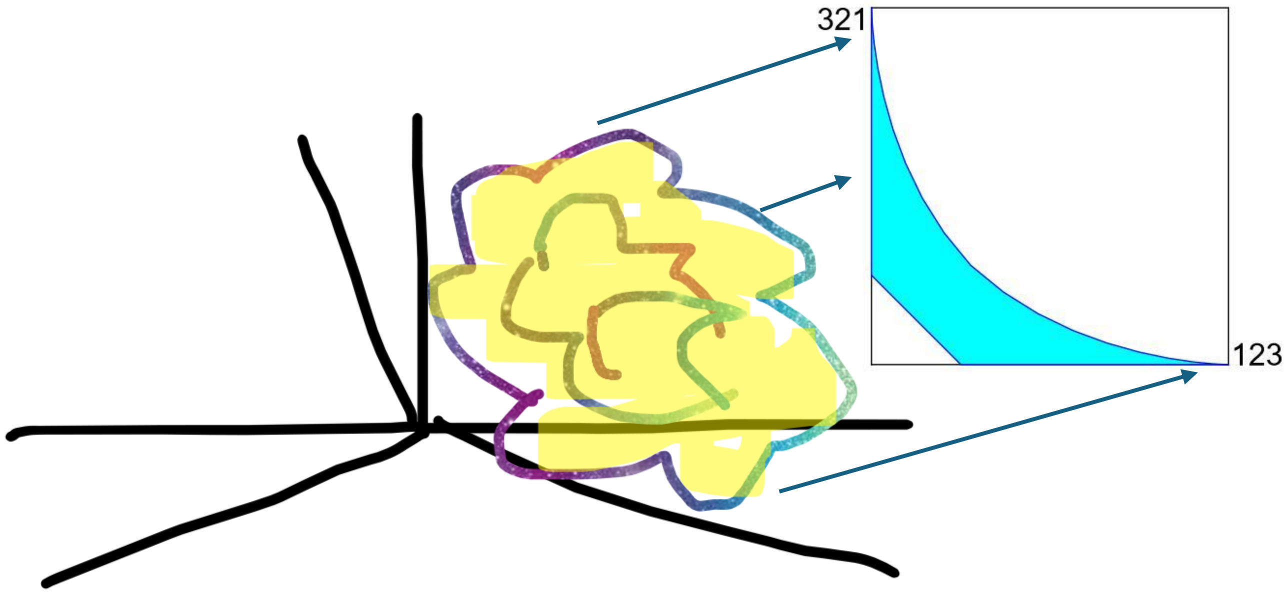


The Great Limit Shape

We can understand the Great Limit Set in terms of its PROJECTIONS and CROSS SECTIONS.

The solution we saw to the $(\delta_{123},\ \delta_{321})$ problem is an example of a projection of the 5-d set onto a 2-d plane.



The Layered Version

Today we will deal with the special case of LAYERED PERMUTATIONS.

A LAYERED PERMUTATION is one that contains no 231 or 312 patterns.

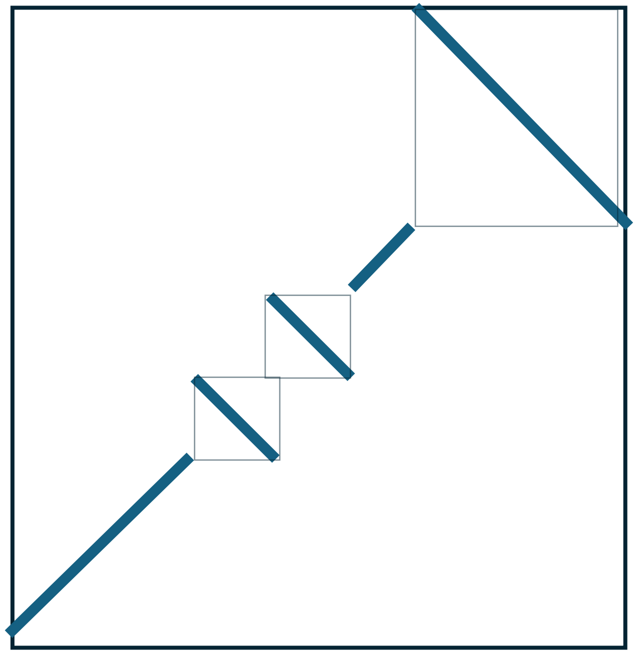
In dealing with layered permutations, we will omit those components of the packing vector, which becomes

$$v = (\delta_{123}, \delta_{132}, \delta_{213}, \delta_{321}) \in R^4.$$

Which of these vectors can be limits of packing vectors of layered permutations?

The answer is a compact subset of R^4, contained in the 3-d standard simplex.

Layered Permutons



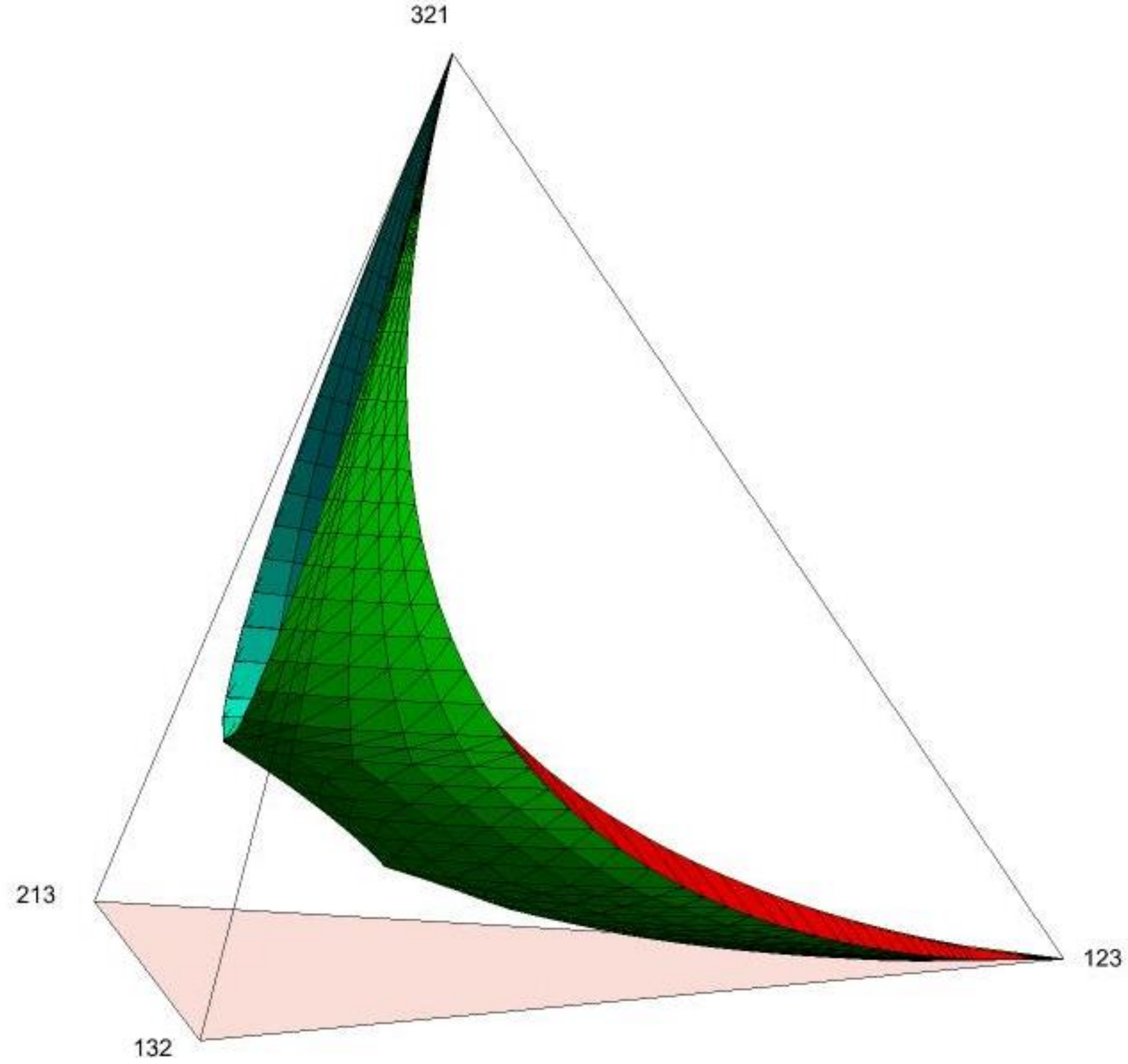
$\langle x_1, x_2, x_3 \rangle$ = sizes of DOWN boxes, in decreasing order. In this case, $\left\langle \frac{1}{3}, \frac{1}{9}, \frac{1}{9} \right\rangle$.

For δ_{123} and δ_{321}, only the *x*'s matter---NOT their order, or anything about the UP boxes.

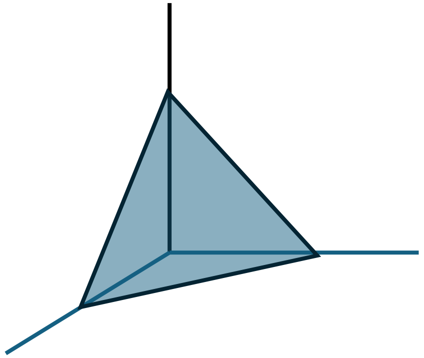
$$\delta_{123} = 1 - 3\sum x_i^2 + 2\sum x_i^3$$
$$\delta_{321} = \sum x_i^3$$

For δ_{132} and δ_{213}, order and placement matter.

Great Limit Shape (layered version)

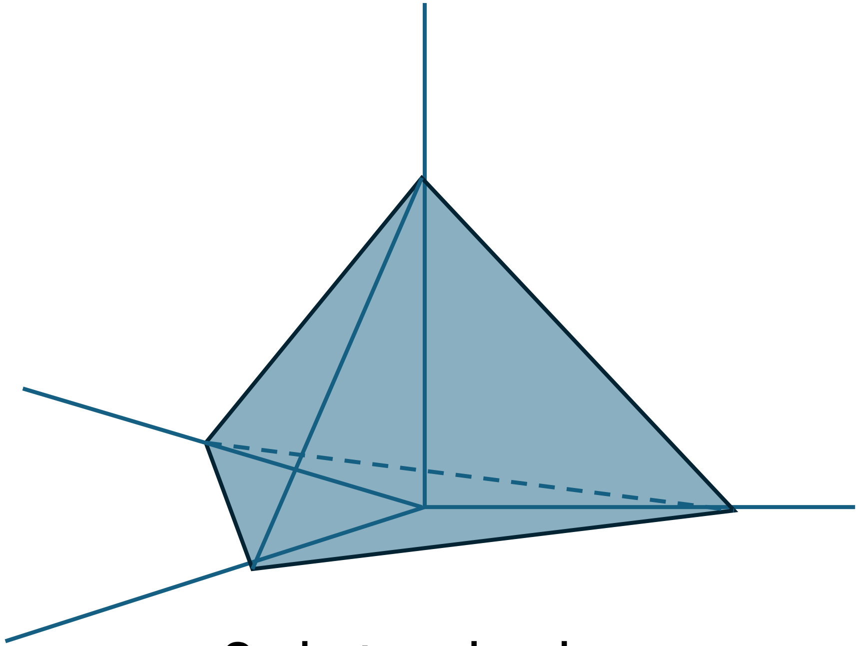


The coordinate system



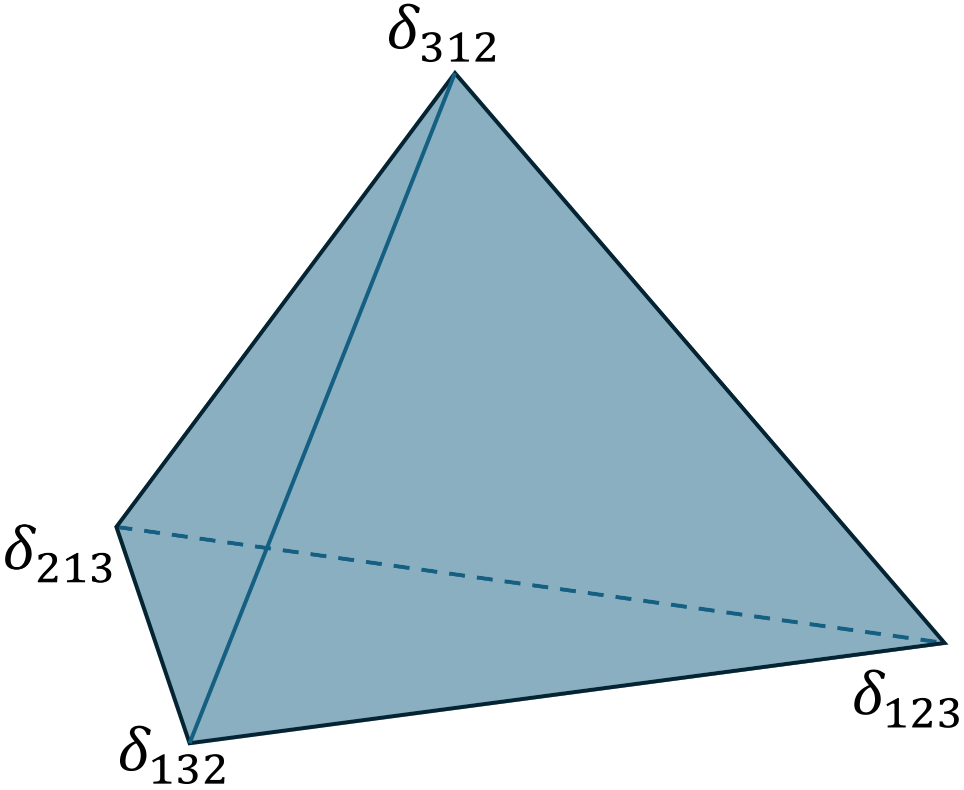
2-d standard simplex in R^3

= triangle



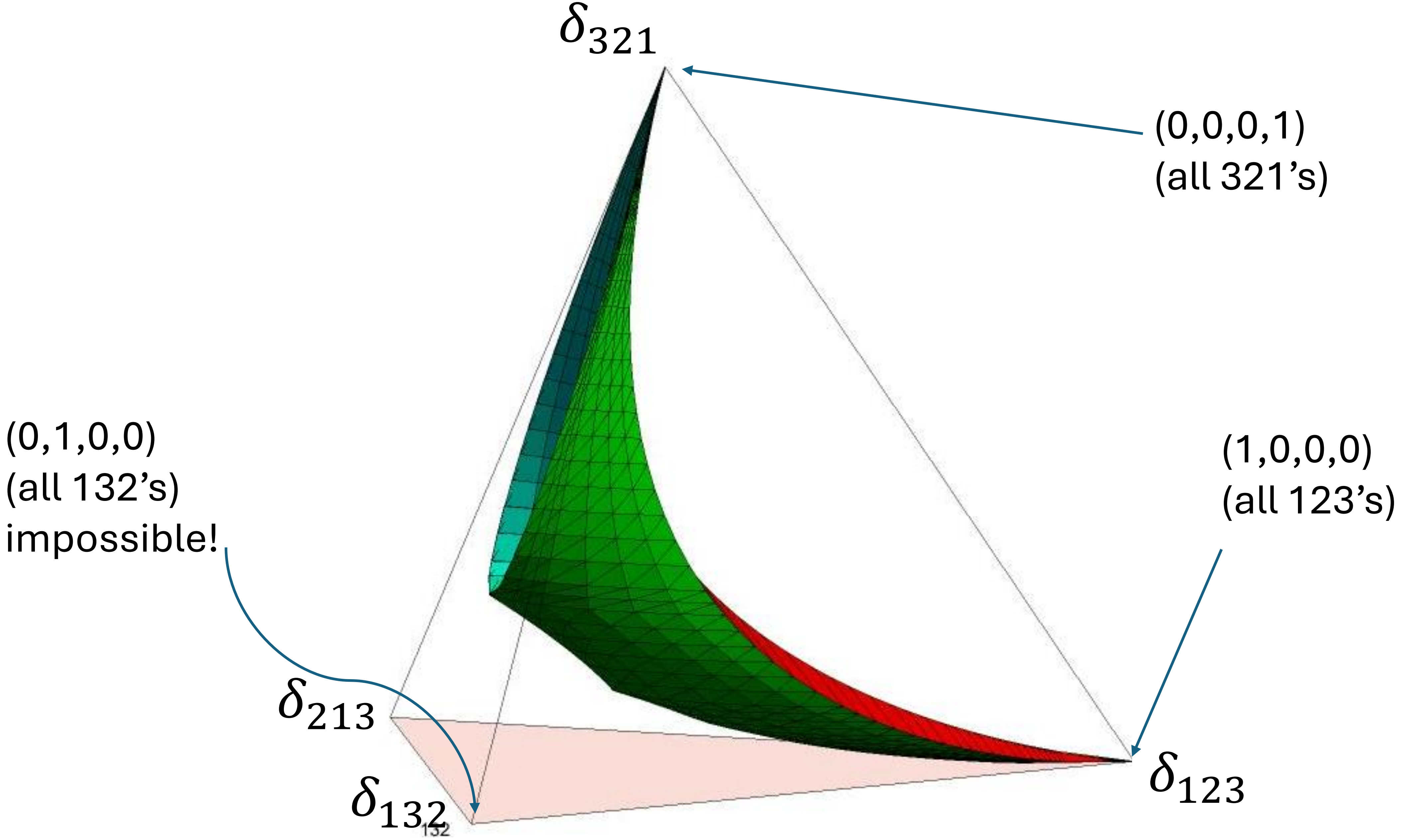
3-d standard simplex in R^4

= tetrahedron



Free-standing tetrahedron
= home of the limiting shape

Great Limit Set (layered version)



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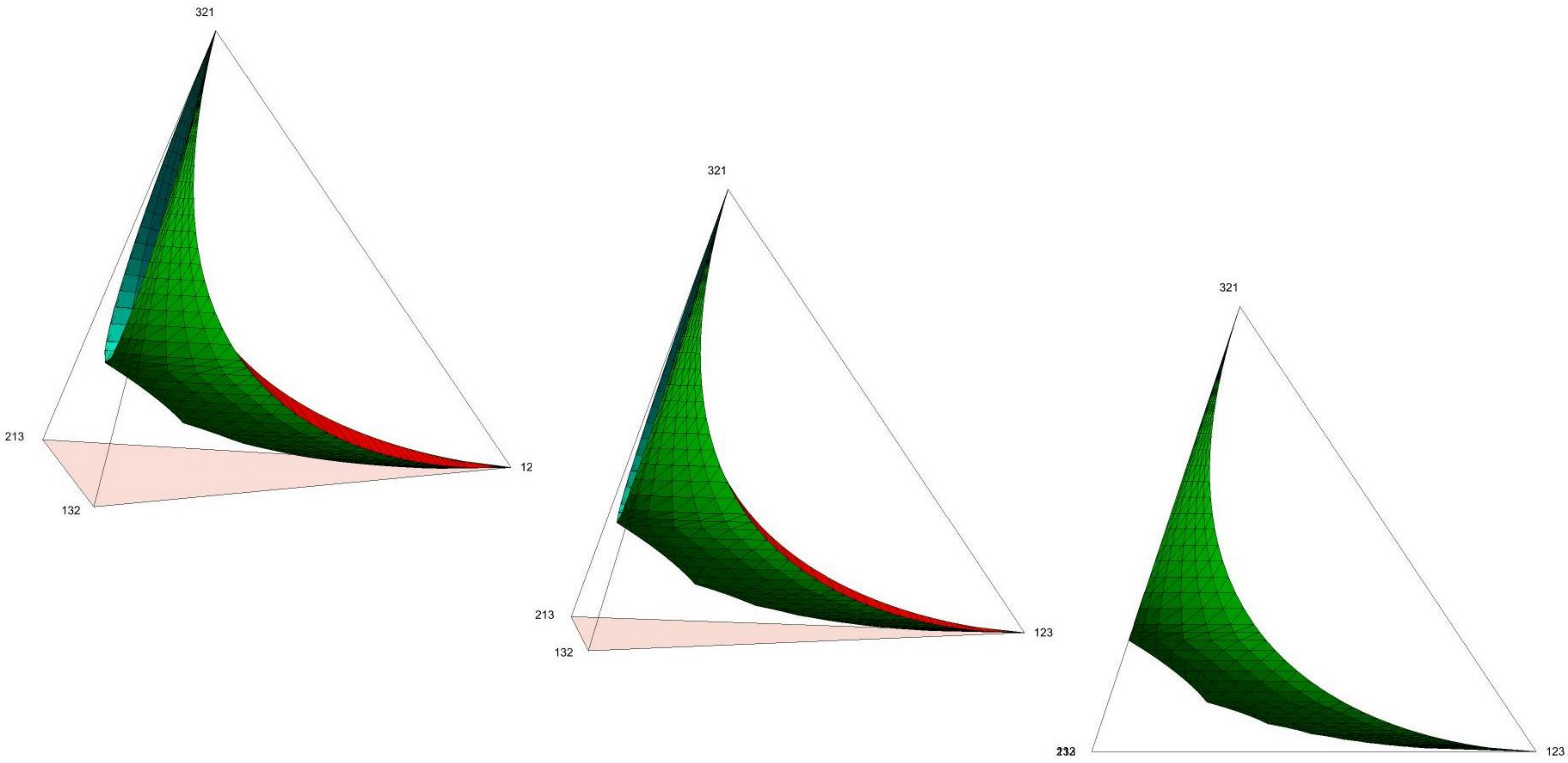
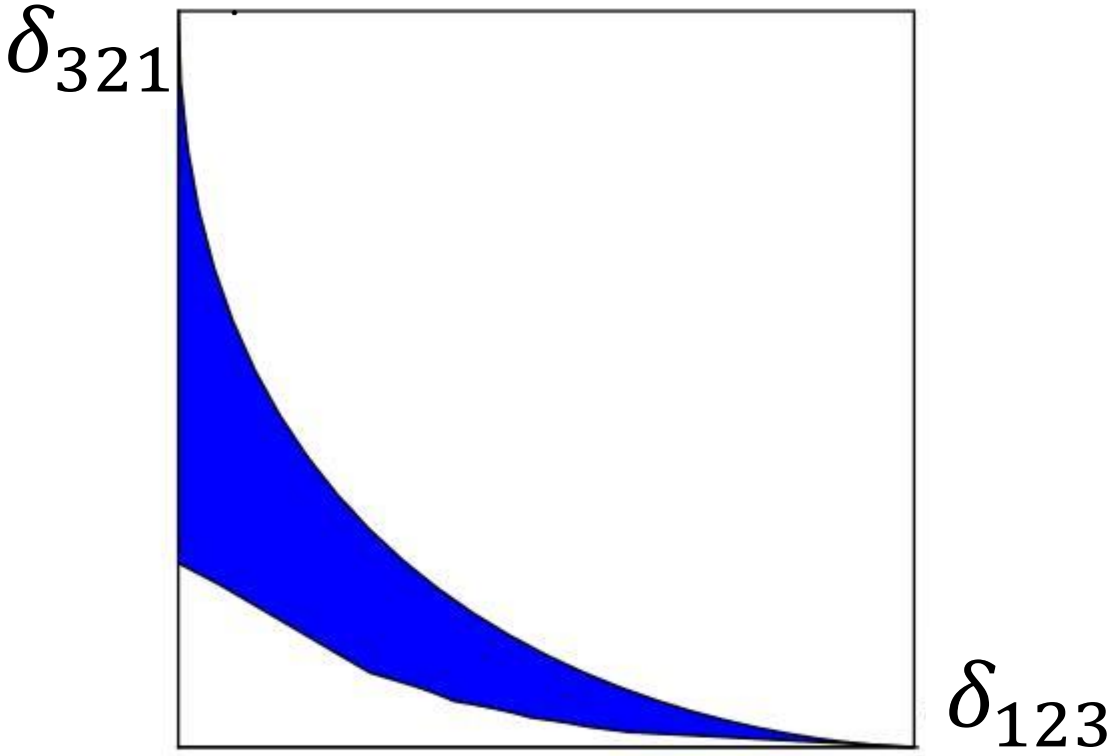
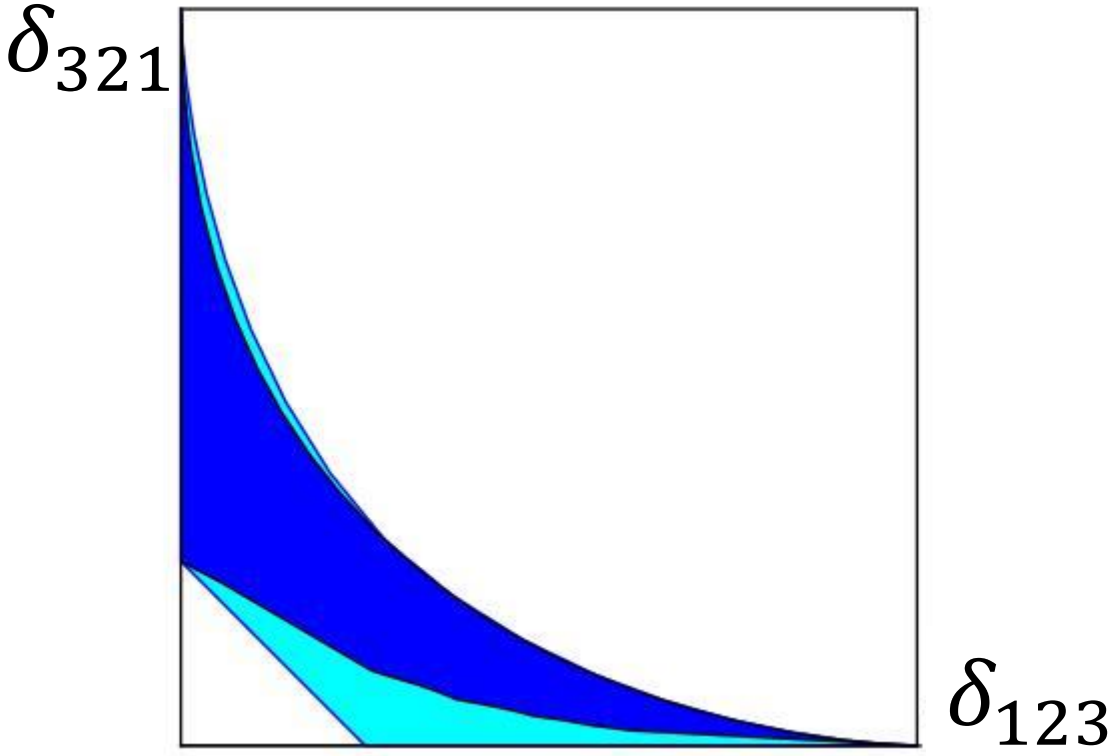

321
213
12
132
321
213
123
132
321
232
123

Layered Version: The Main Diagram

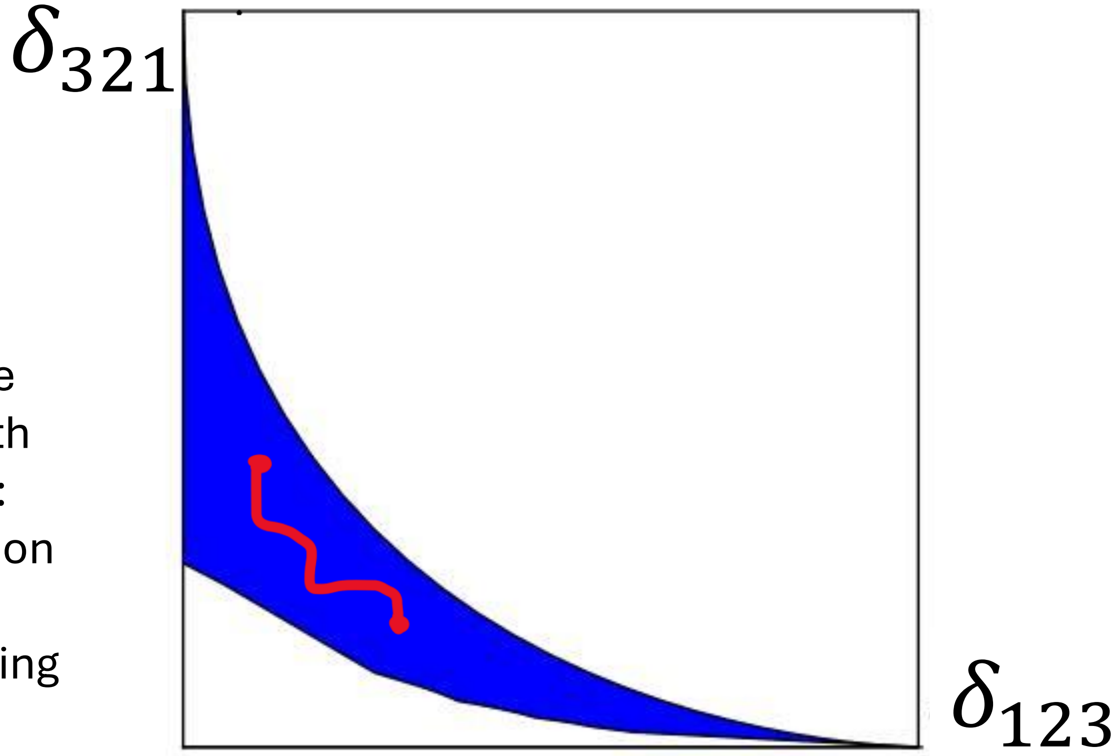


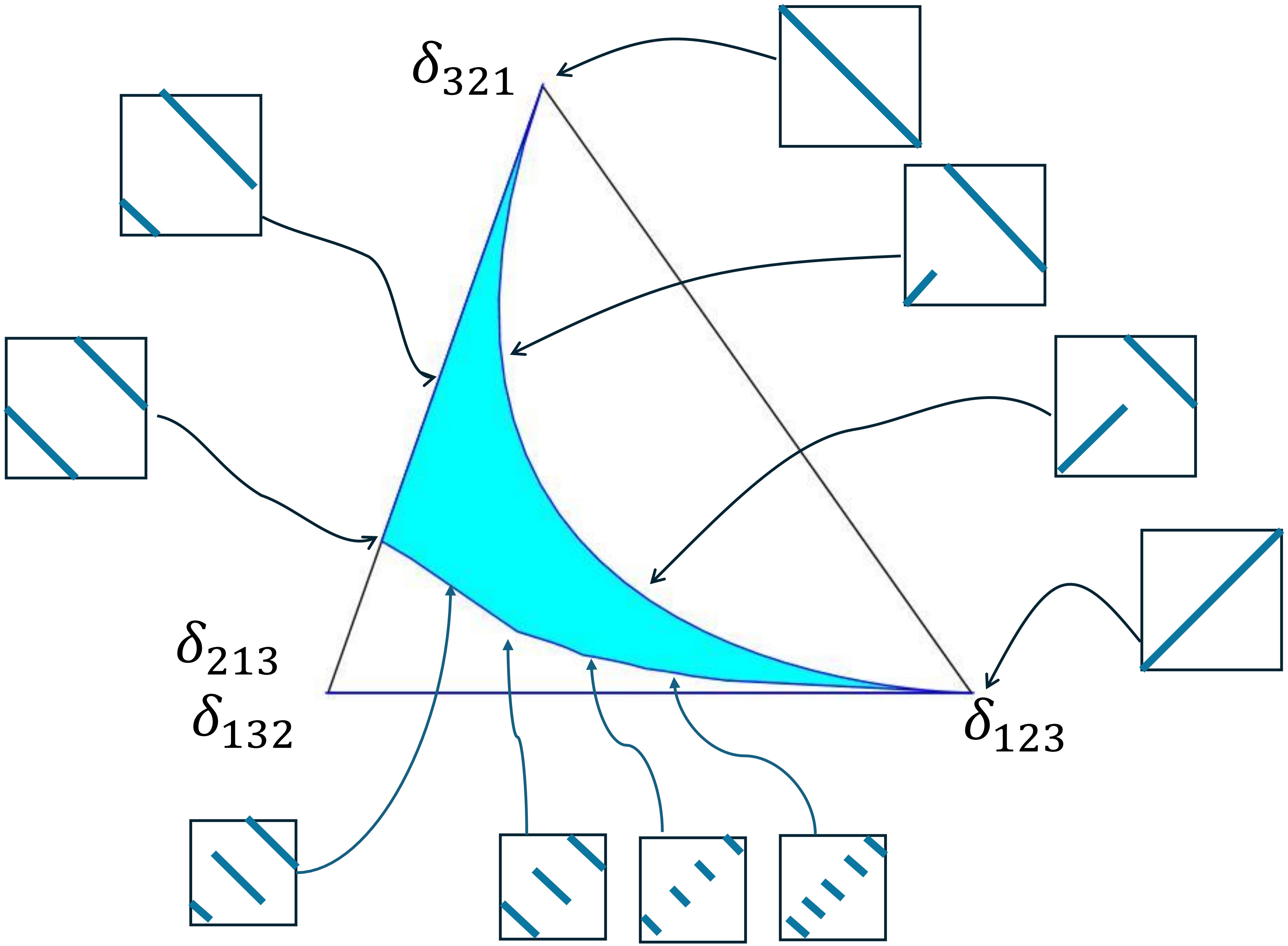
Layered Version: The Main Diagram



Layered Version: The Main Diagram

Every path in this set is the image of a continuous path among permutons. Proof: Each point in the blue region corresponds to a unique permuton whose decreasing layers have sizes
w, wz, …, wz, (1-nw)z
for w, z in [0,1].





δ_{321}
δ_{213}
δ_{132}
δ_{123}

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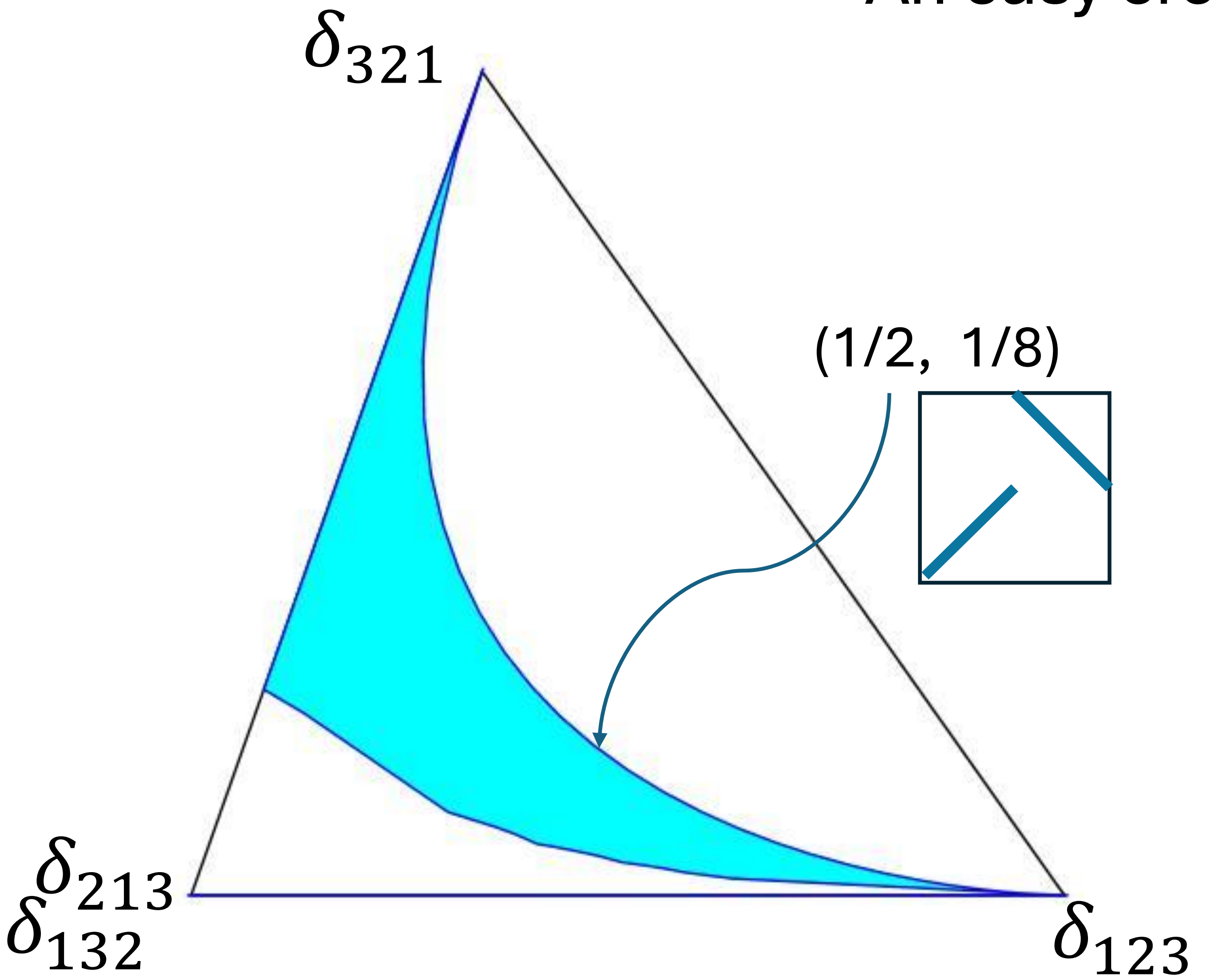
The Cross Sections

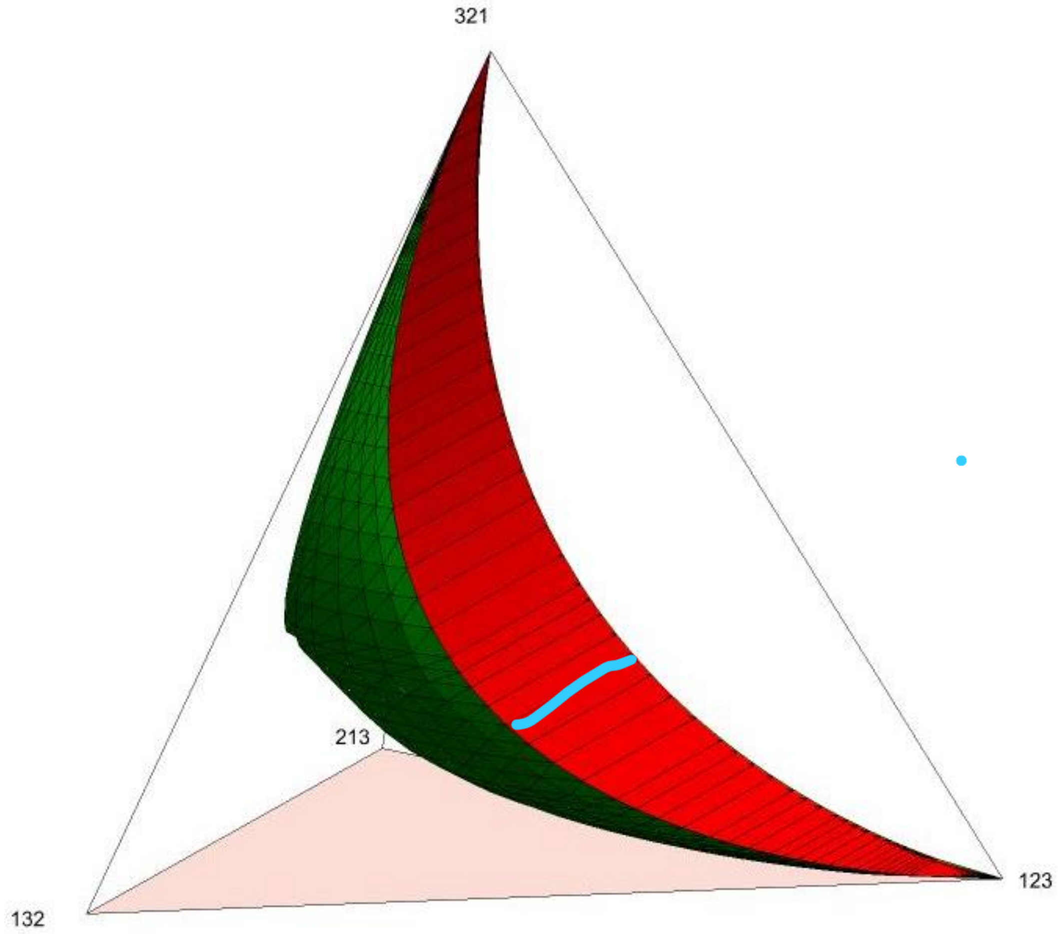
If we knew --- for every point in the main diagram --- what are the possible values of $(\delta_{132}, \delta_{213})$, then we would know the entire shape.

Because $\delta_{213} = 1 - \delta_{123} - \delta_{321} - \delta_{132}$, and both δ_{123} and δ_{321} are given by the point we have chosen, we just need to know the possible values of δ_{132}.

Start with points on the boundary of the main diagram.

An easy cross section

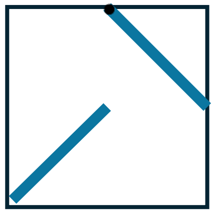




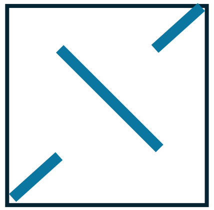
The cross section at (1/2,1/8)

Suppose $\delta_{123} = 1/2$ and $\delta_{321} = 1/8.$ Those values define a point on the upper boundary of the main diagram. They leave 3/8 for δ_{132} and δ_{213} .

Fact: δ_{132} can take any value in [0, 3/8].



has packing vector (1/2, 3/8, 0, 1/8).

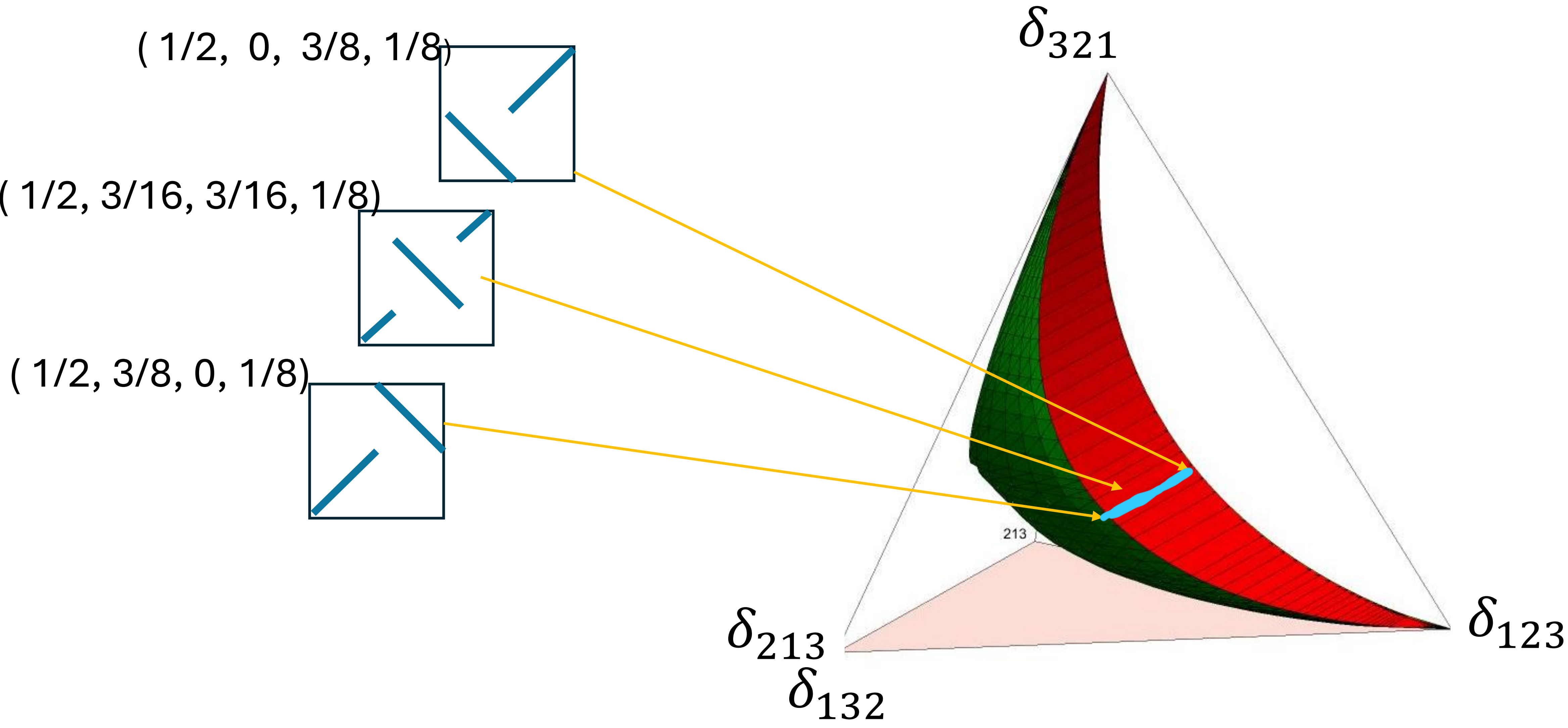


has packing vector (1/2, 3/16, 3/16, 1/8).

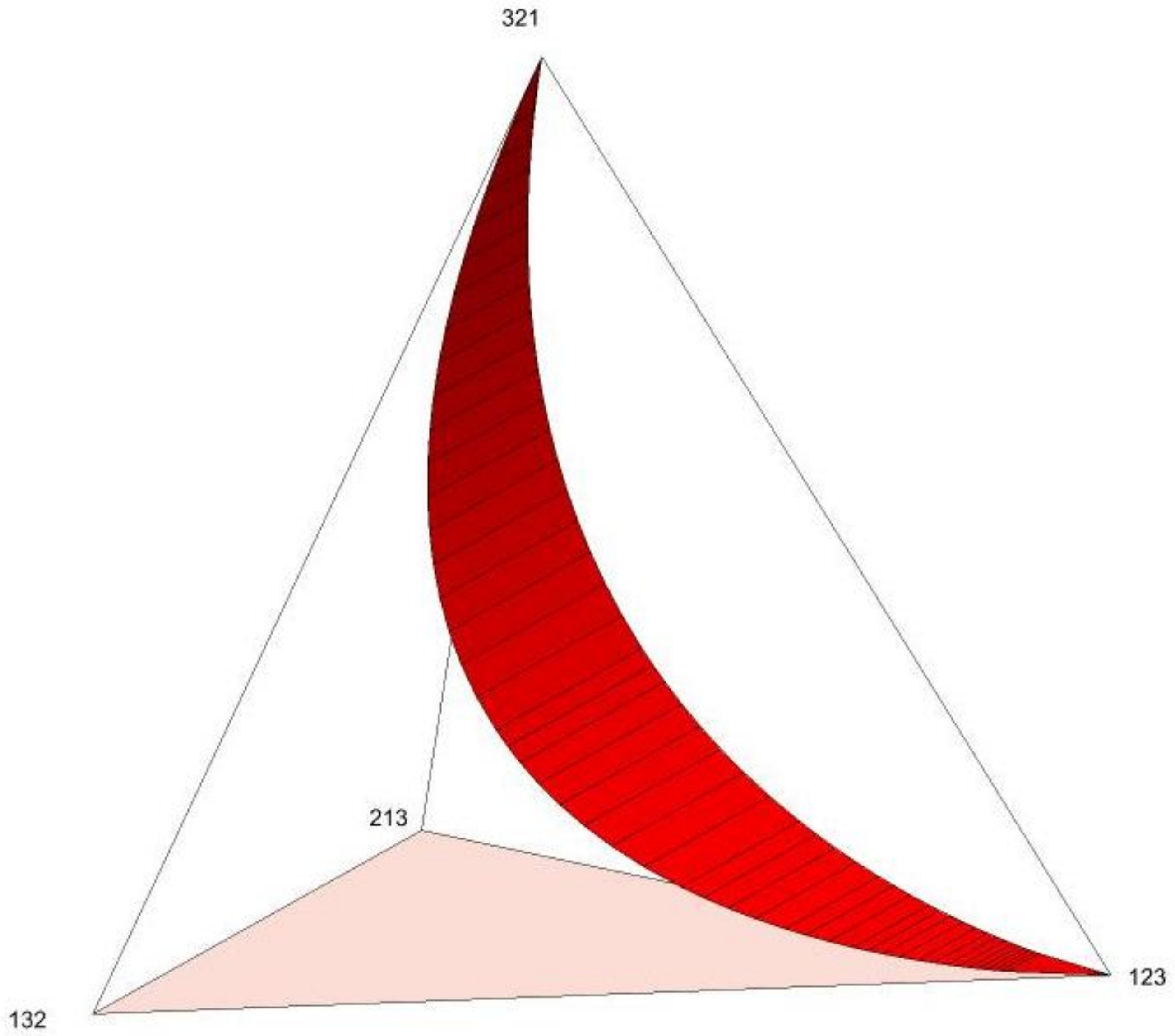
has packing vector (1/2, 0, 3/8, 1/8).

The “up” bits give us slack to adjust δ_{132} to any value we like.

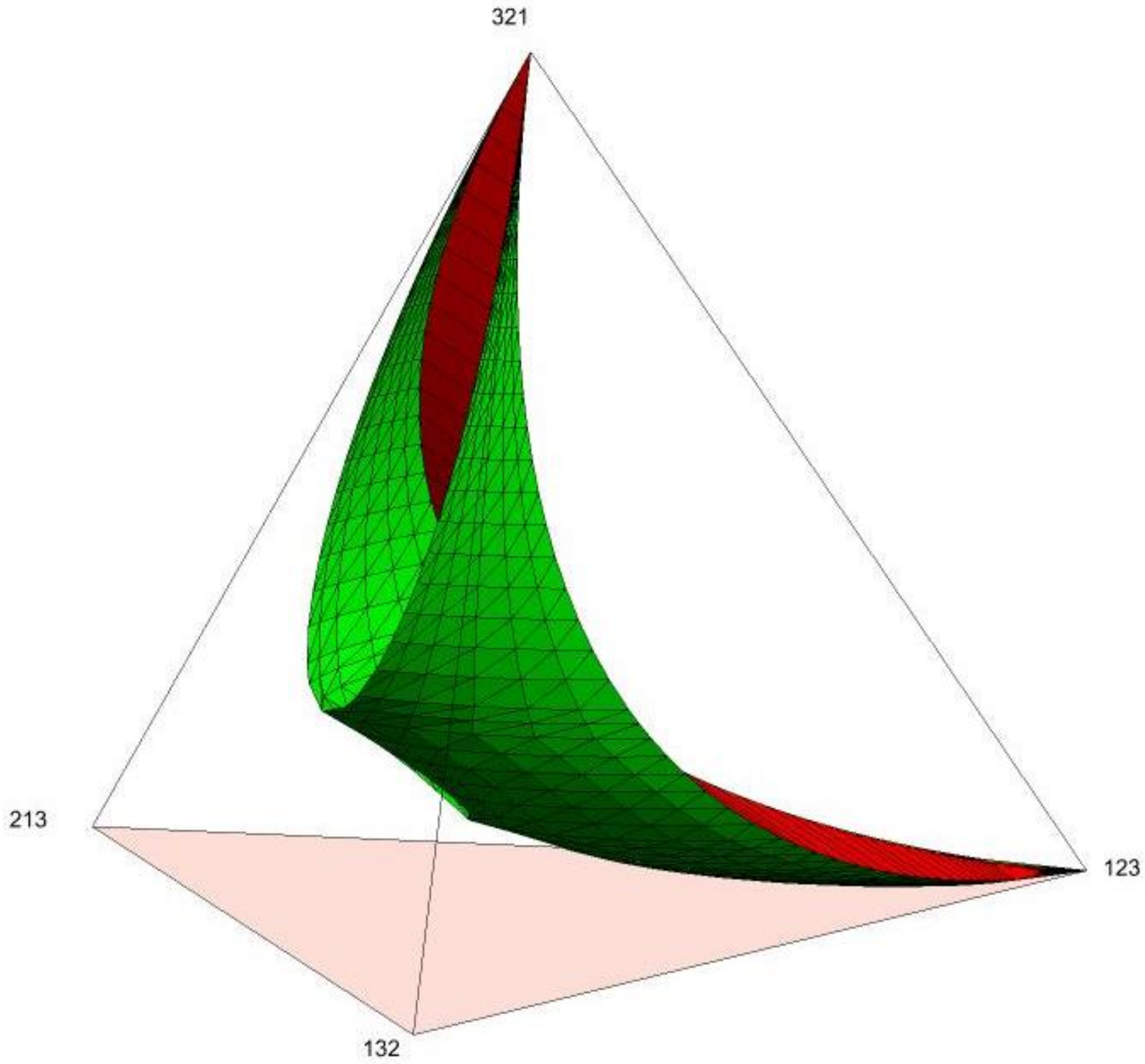
An easy cross section

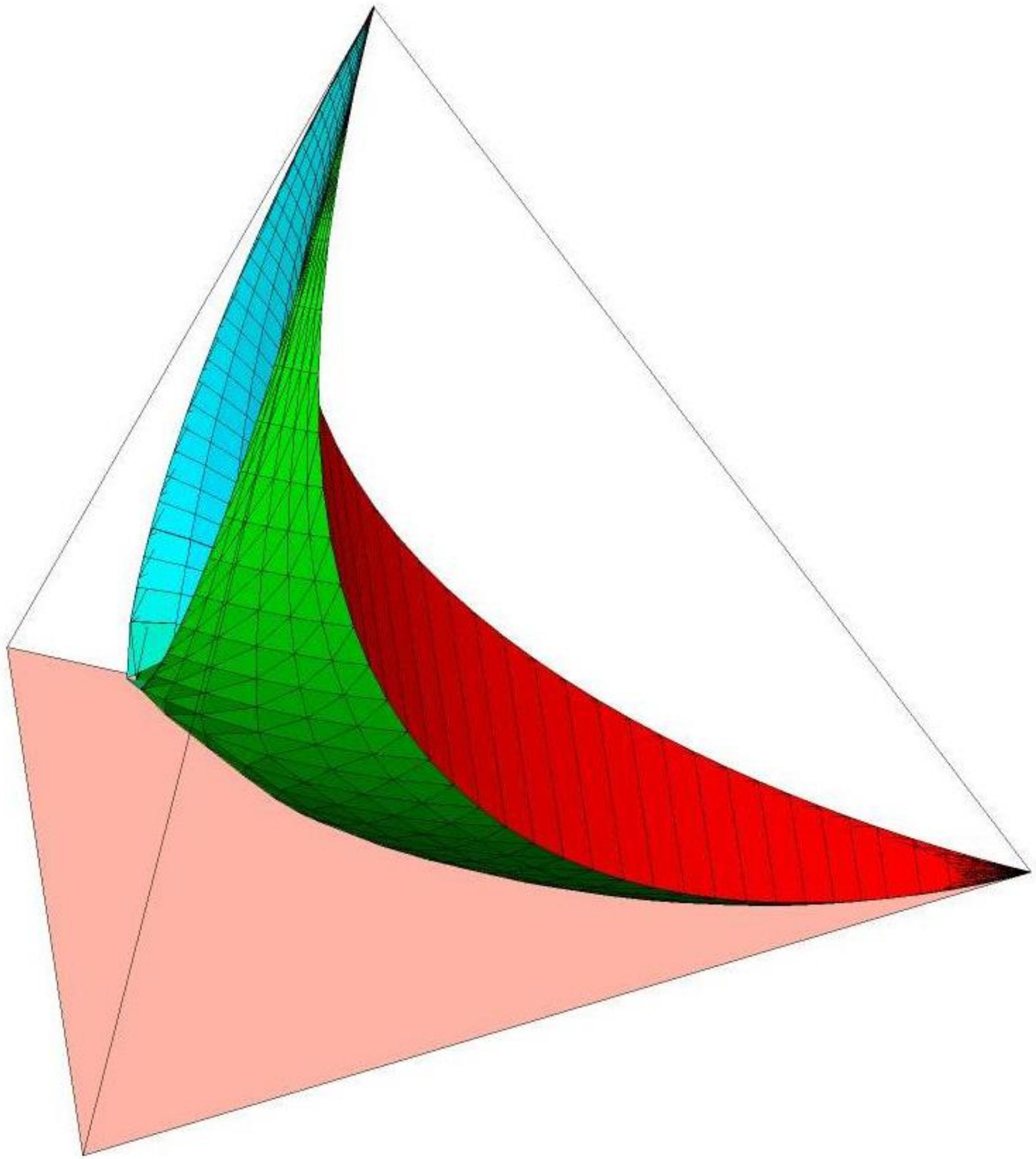


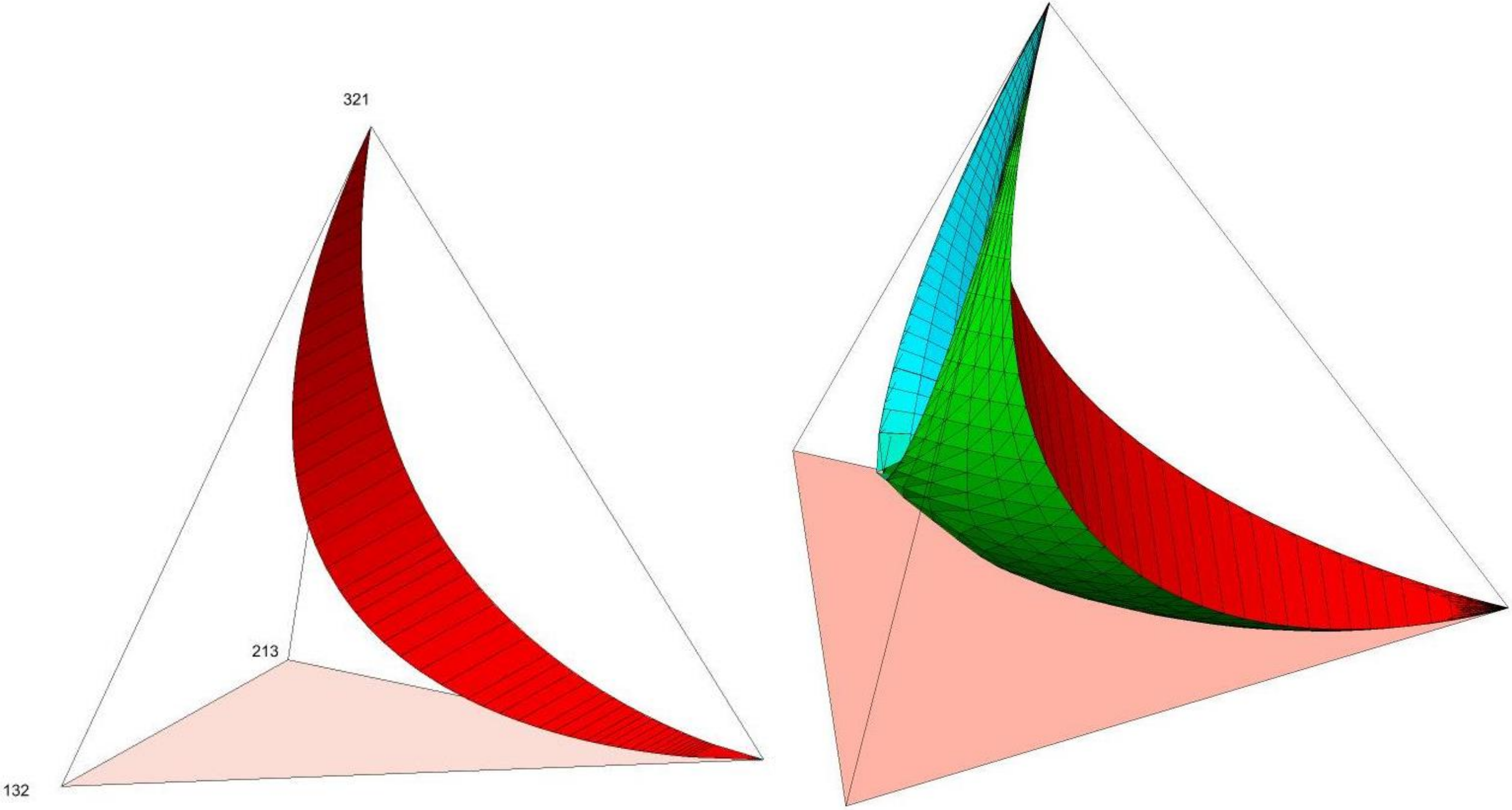
The top surface: Just the easy cross sections



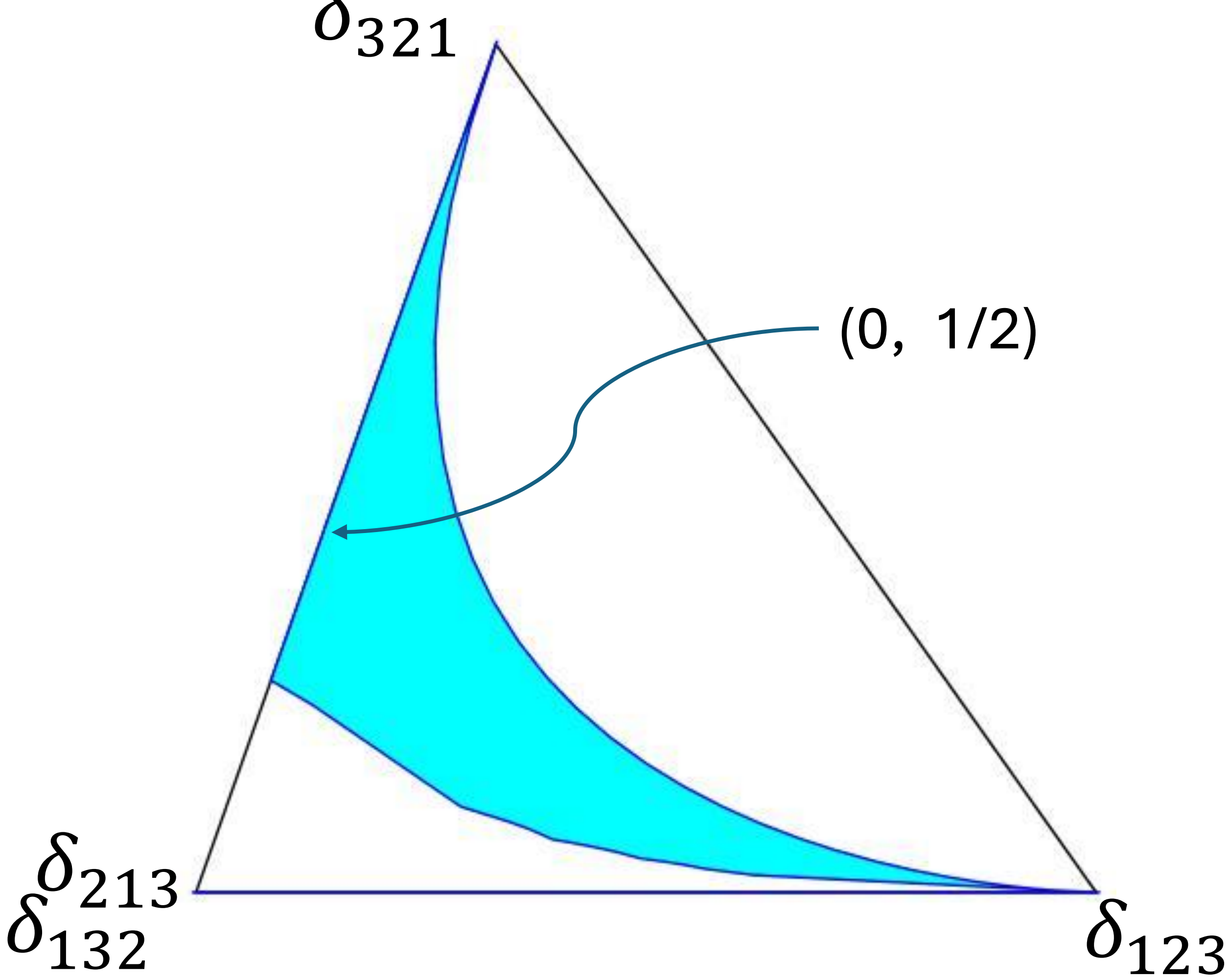
Top and Sides

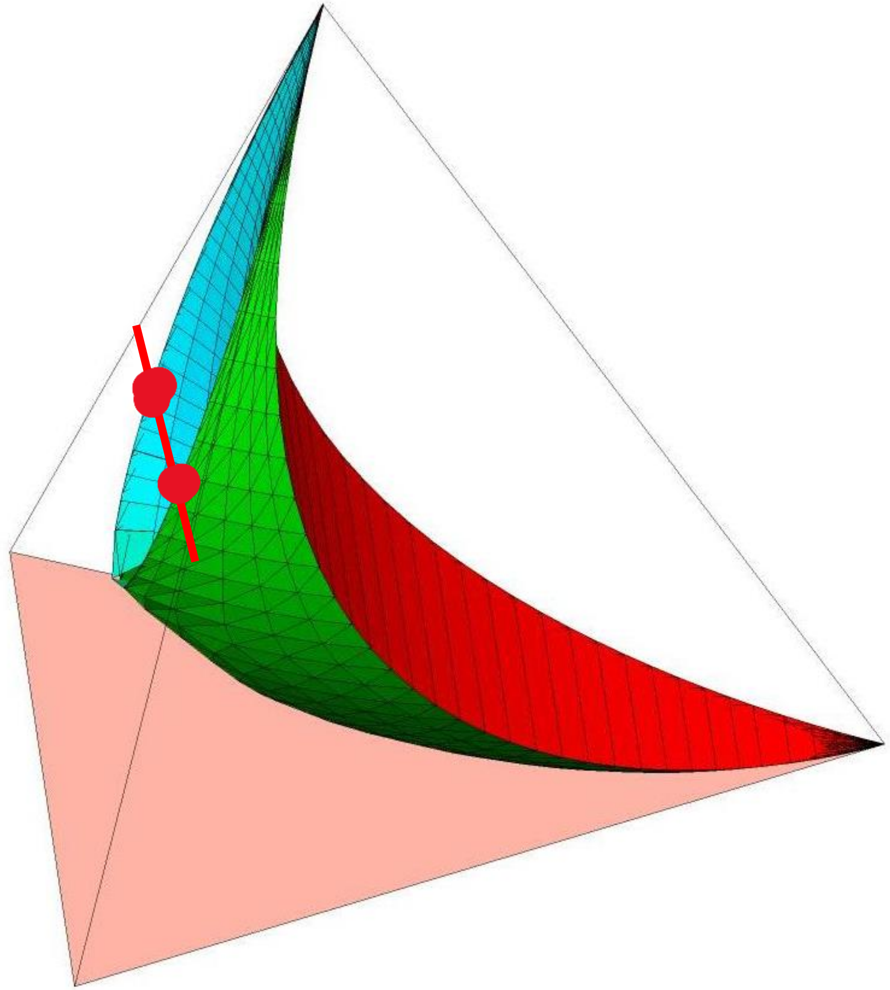




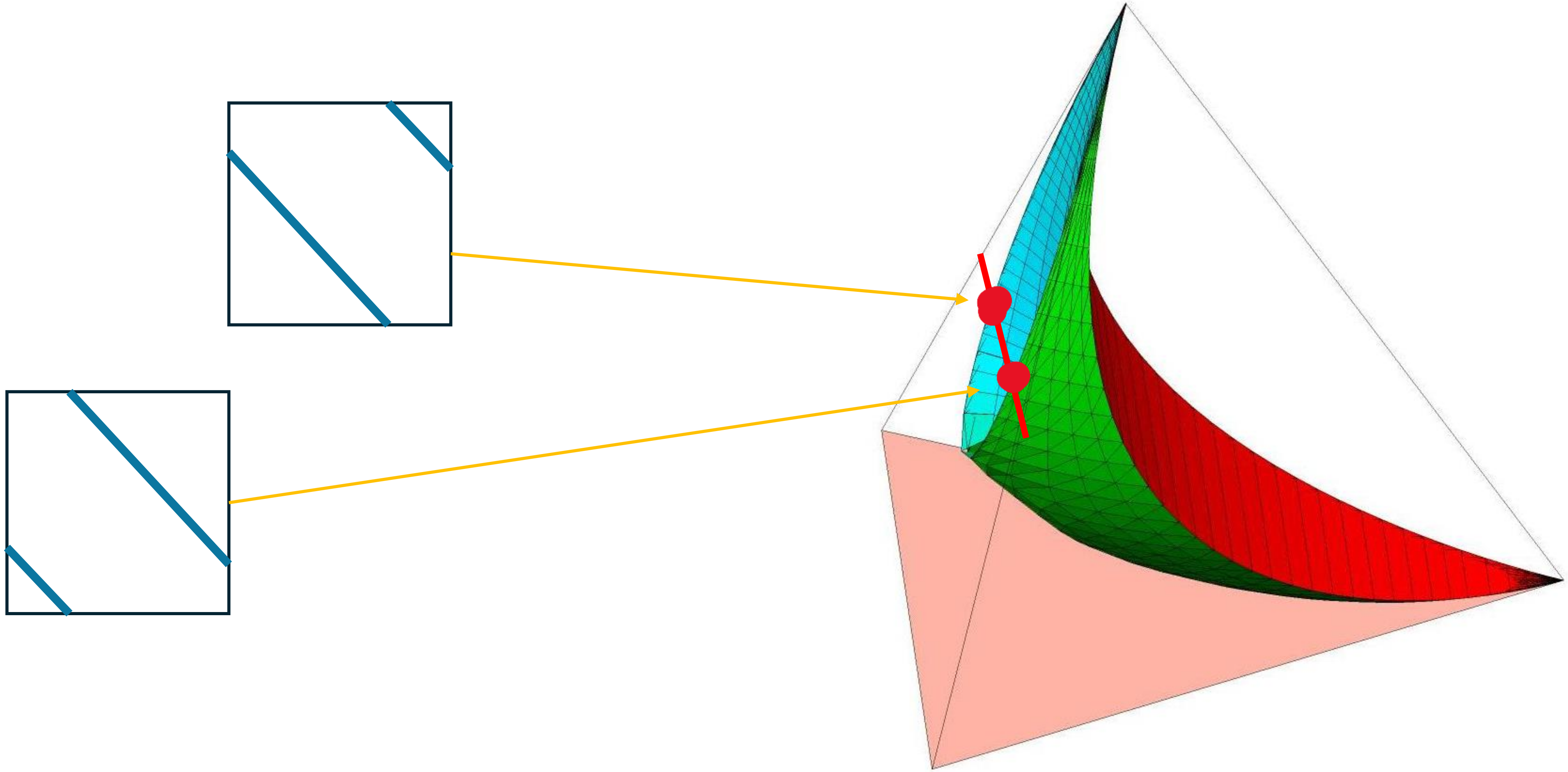

321
213
132

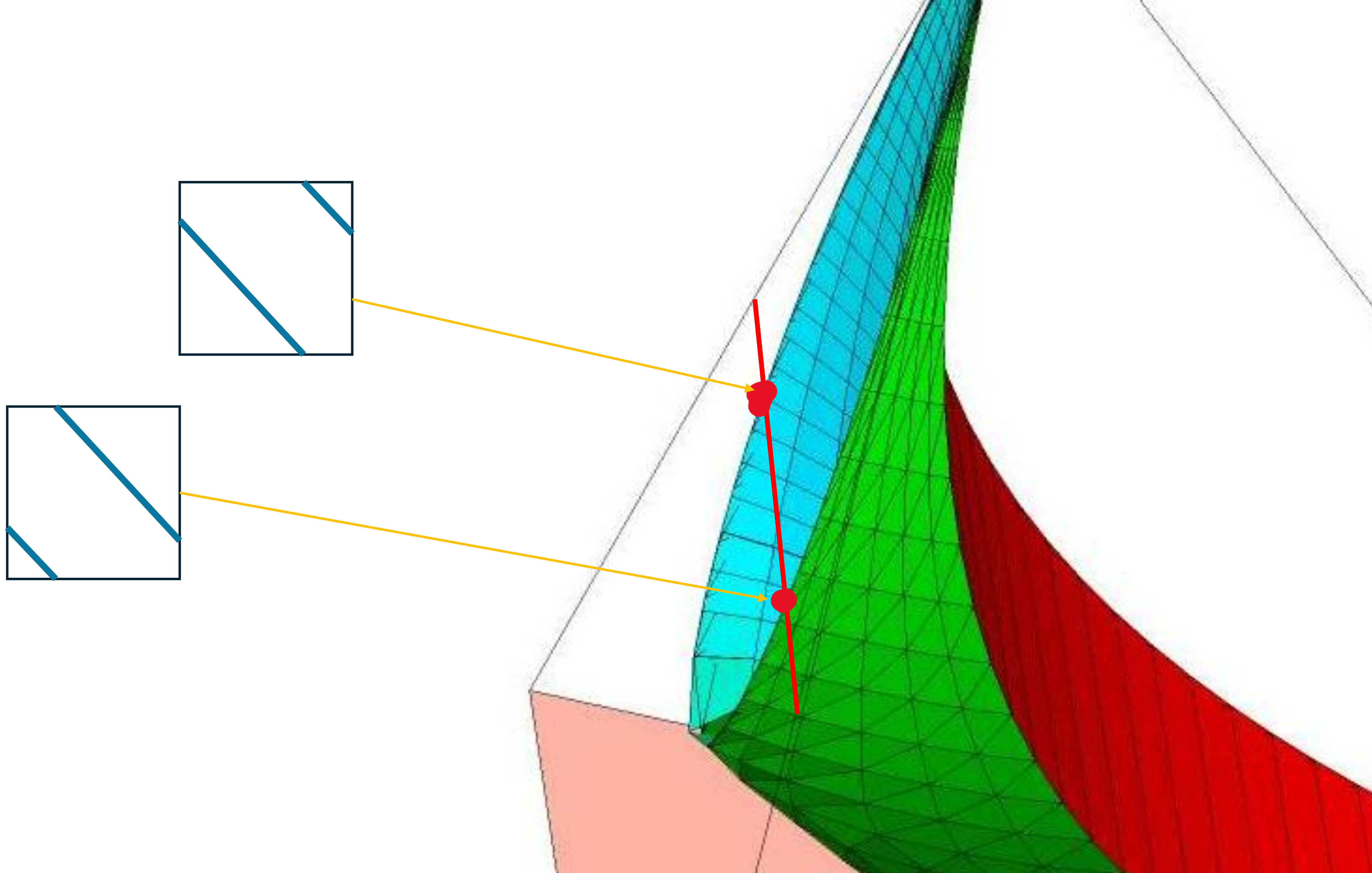
A harder cross section



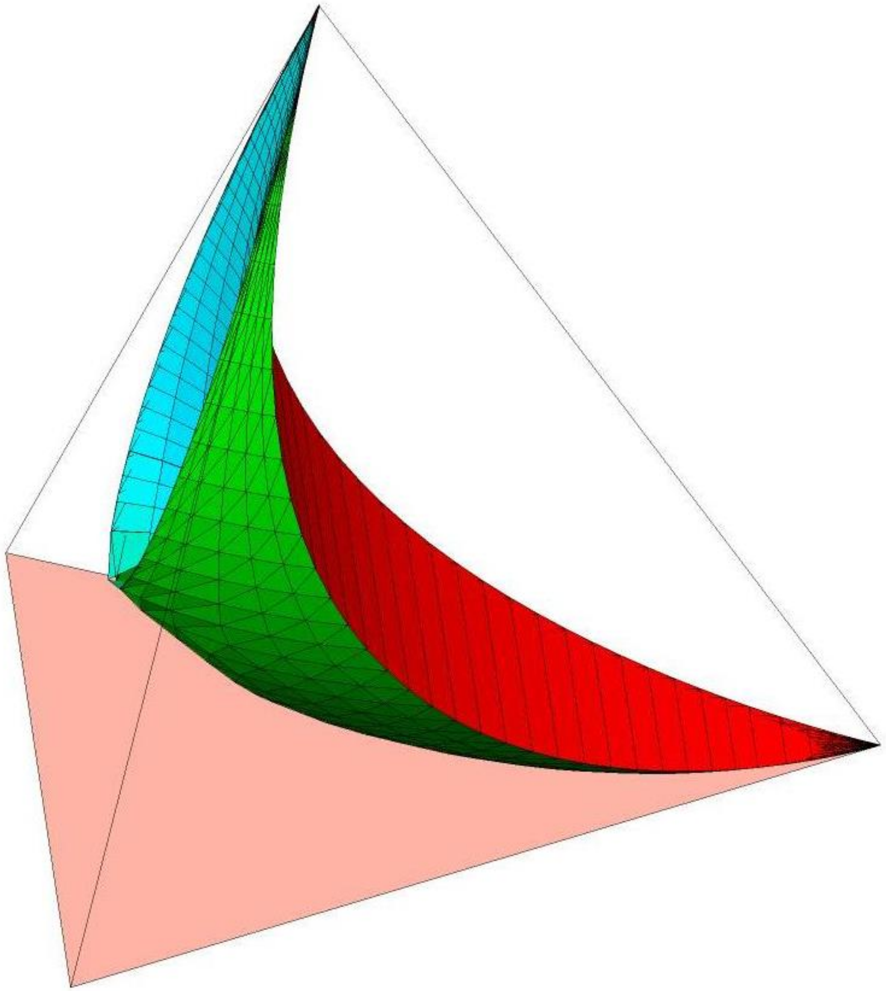


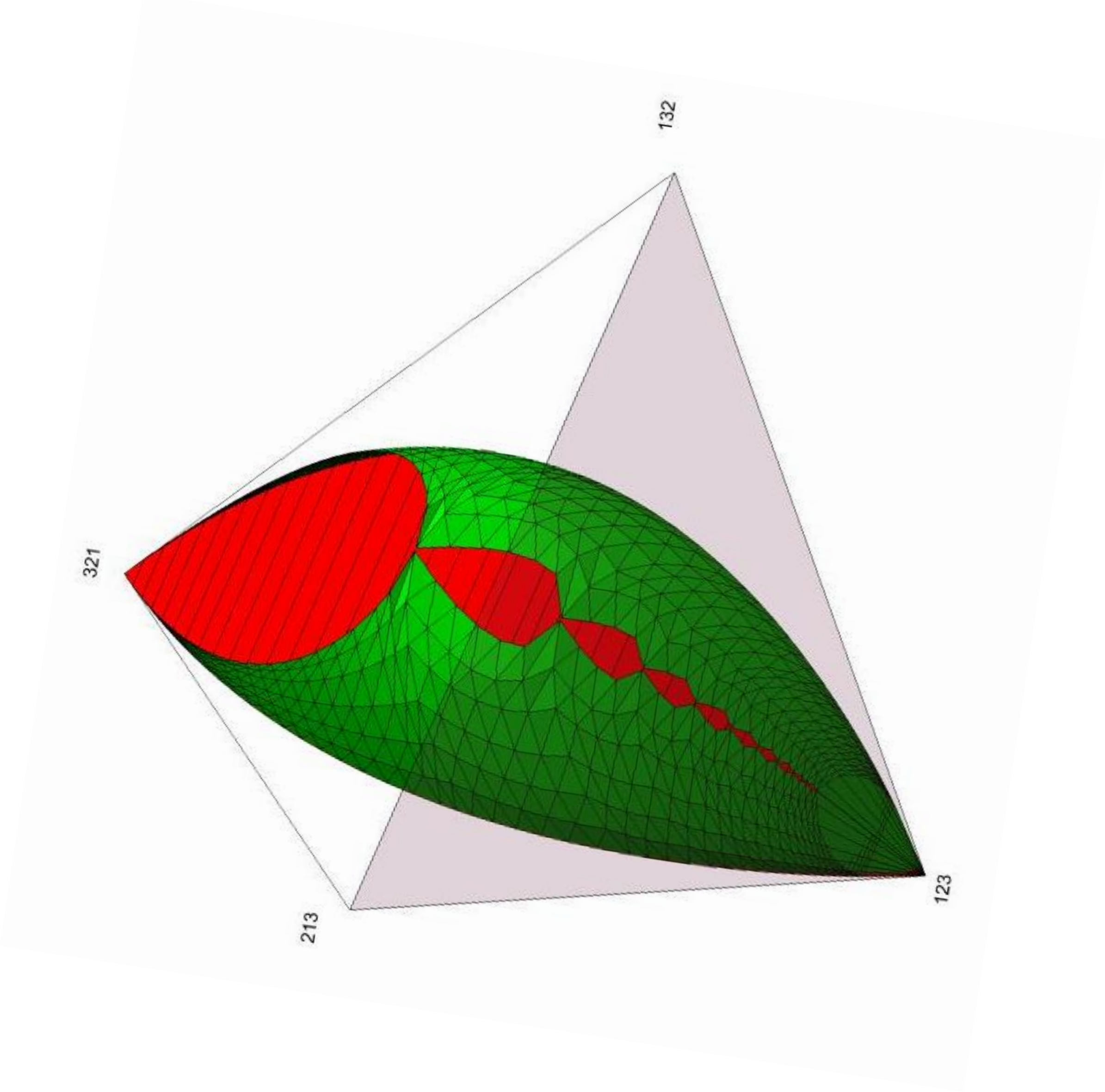
A harder cross section



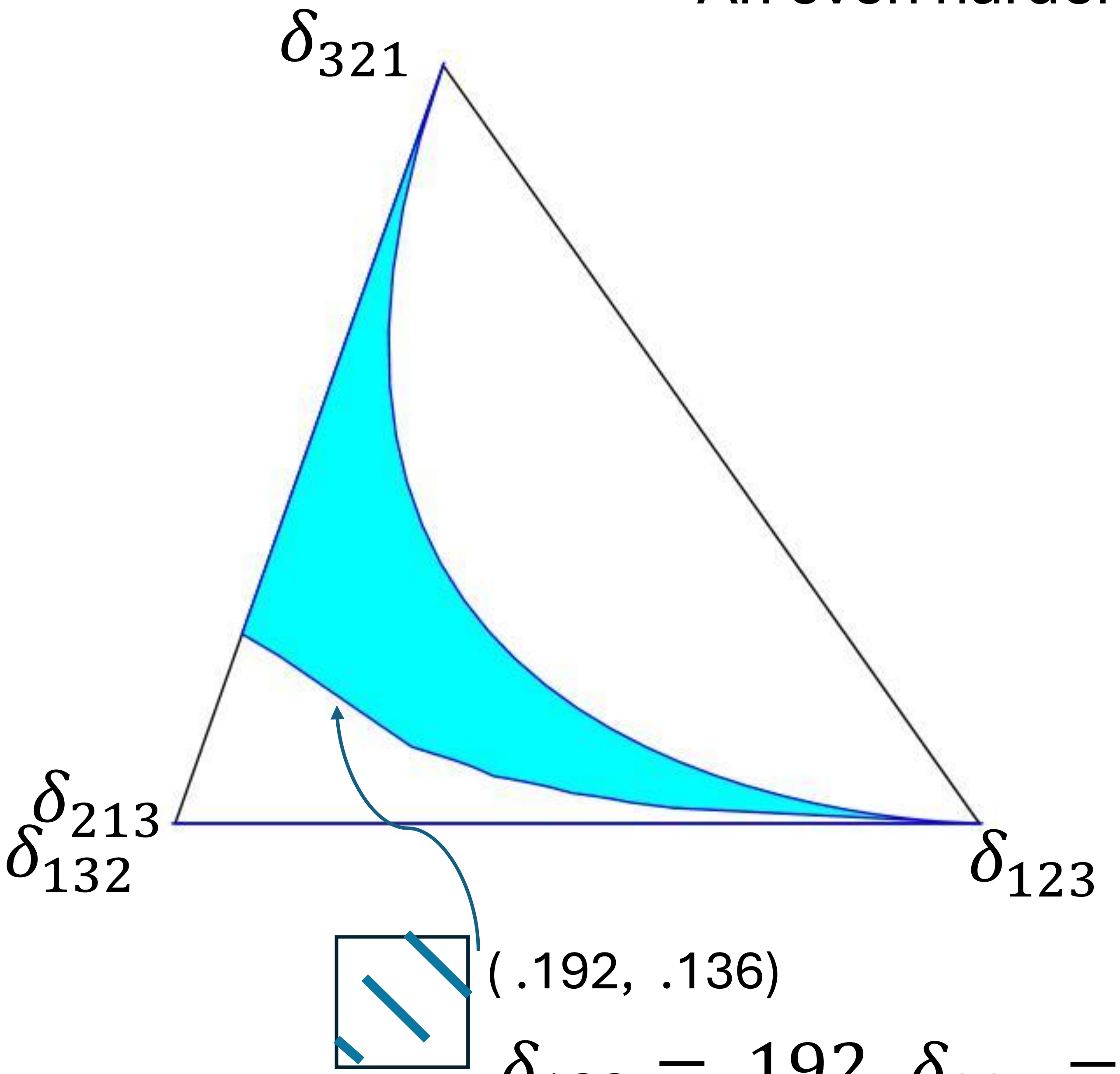


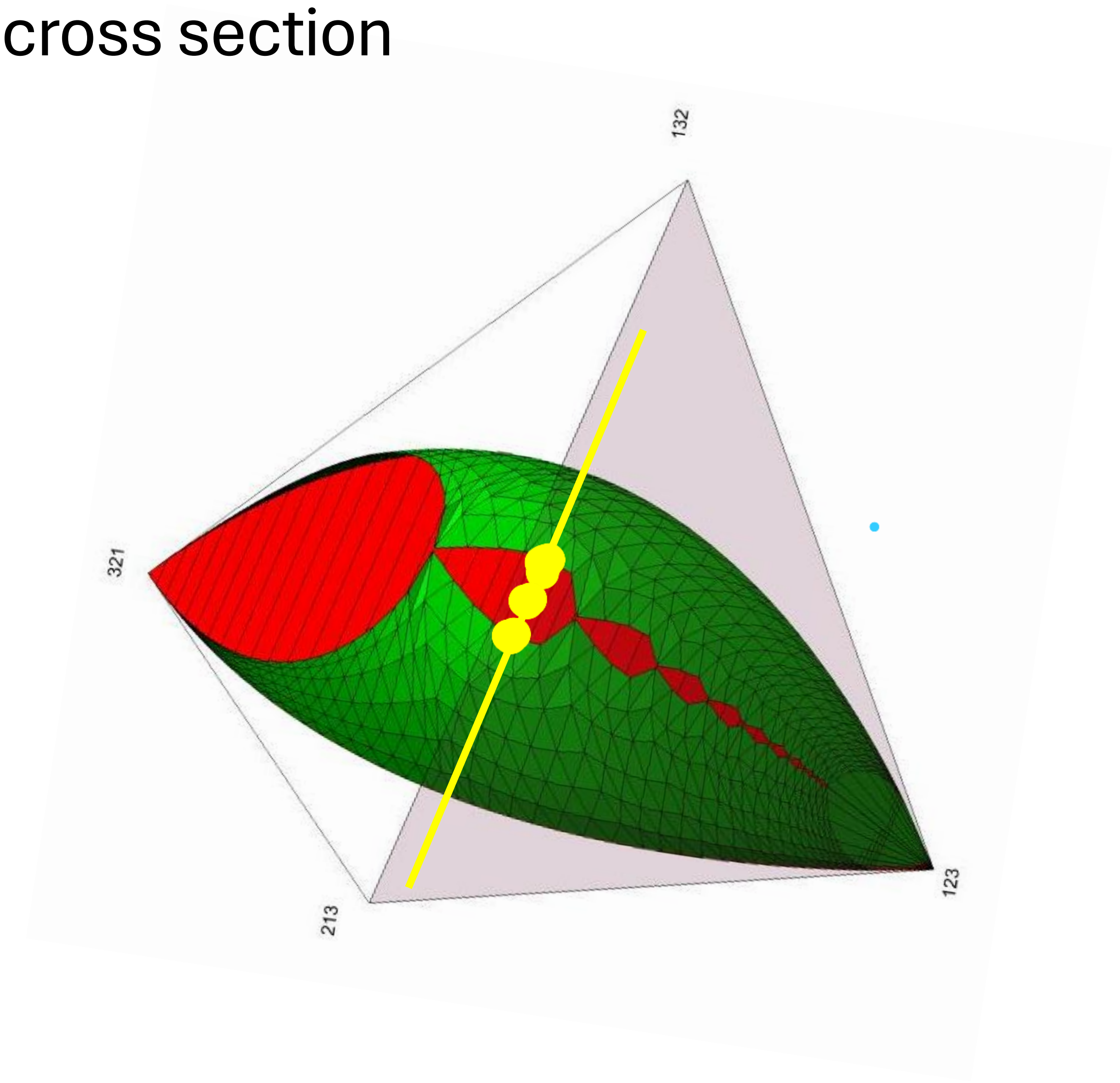
Flip it over…



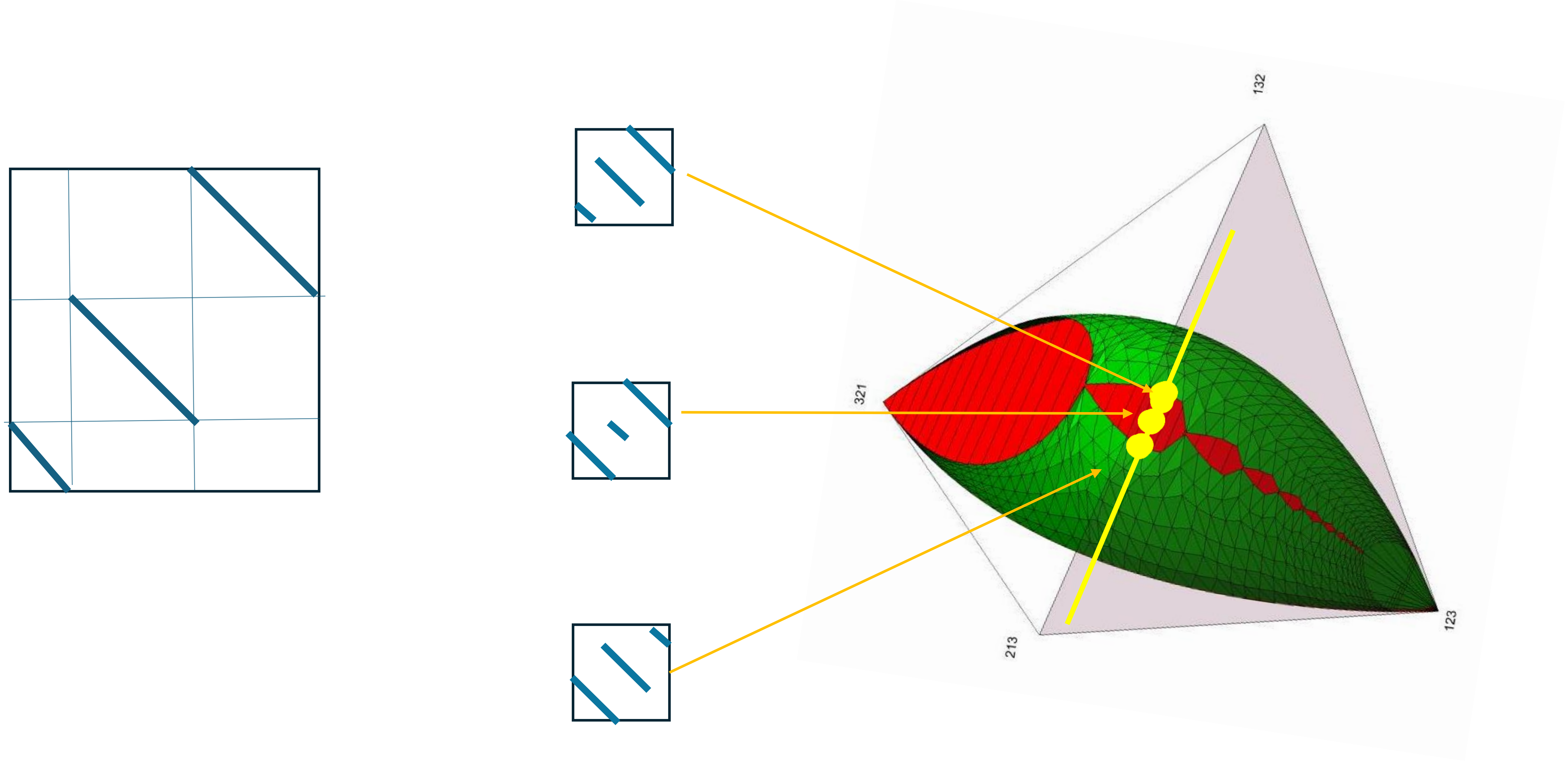


An even harder cross section





$$\delta_{123} = .192,\ \delta_{321} = .136$$


132
321
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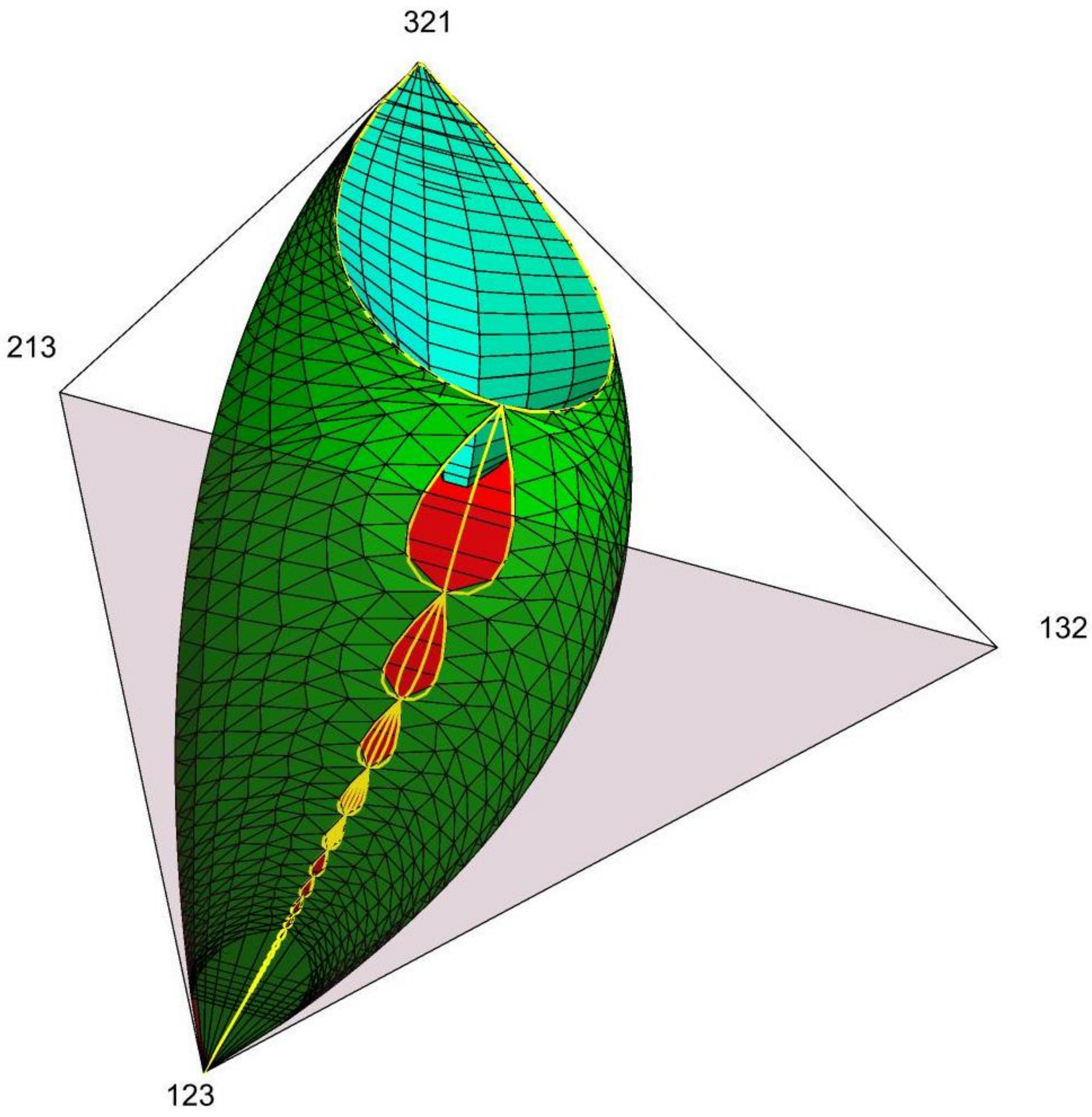
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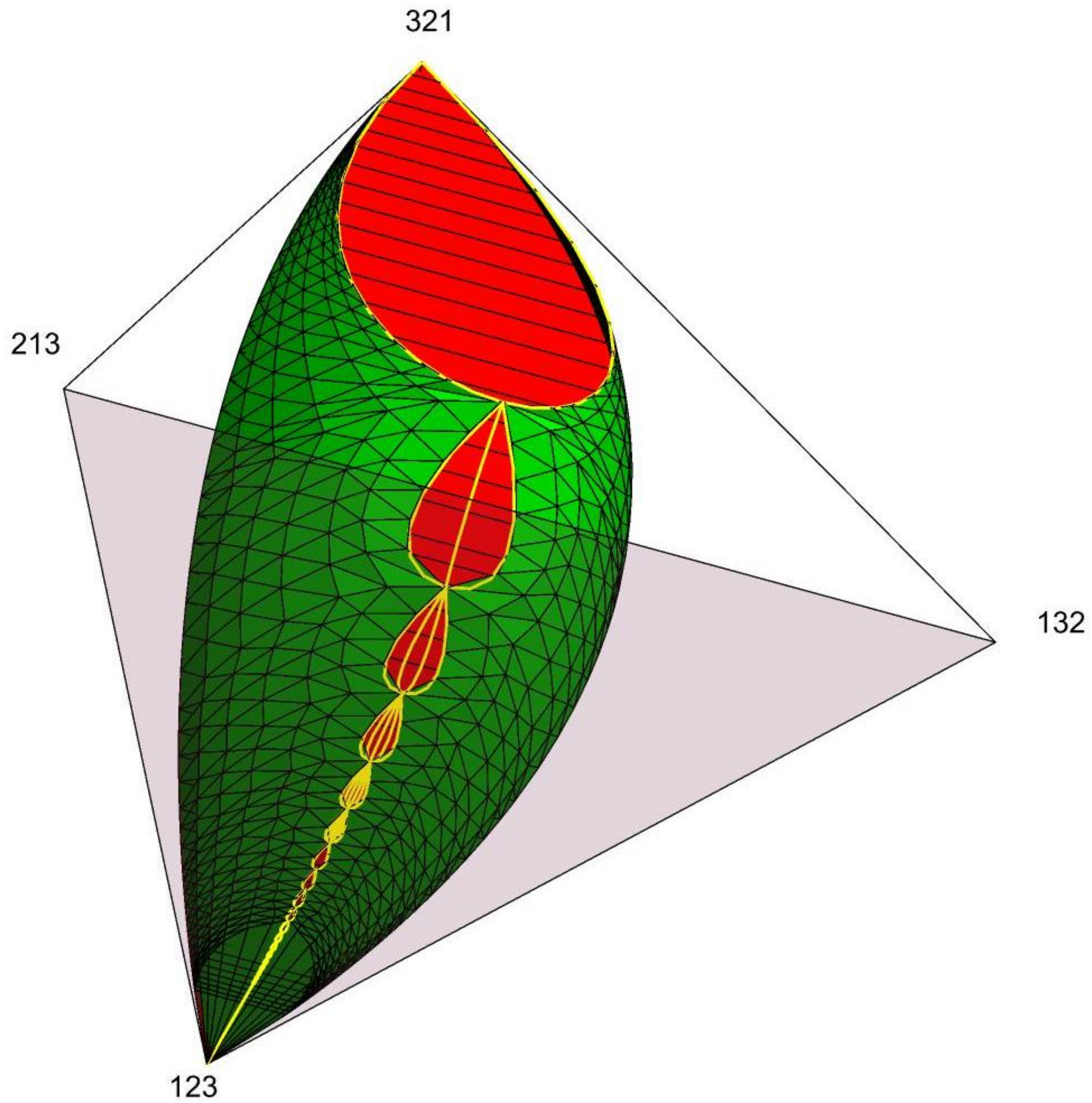

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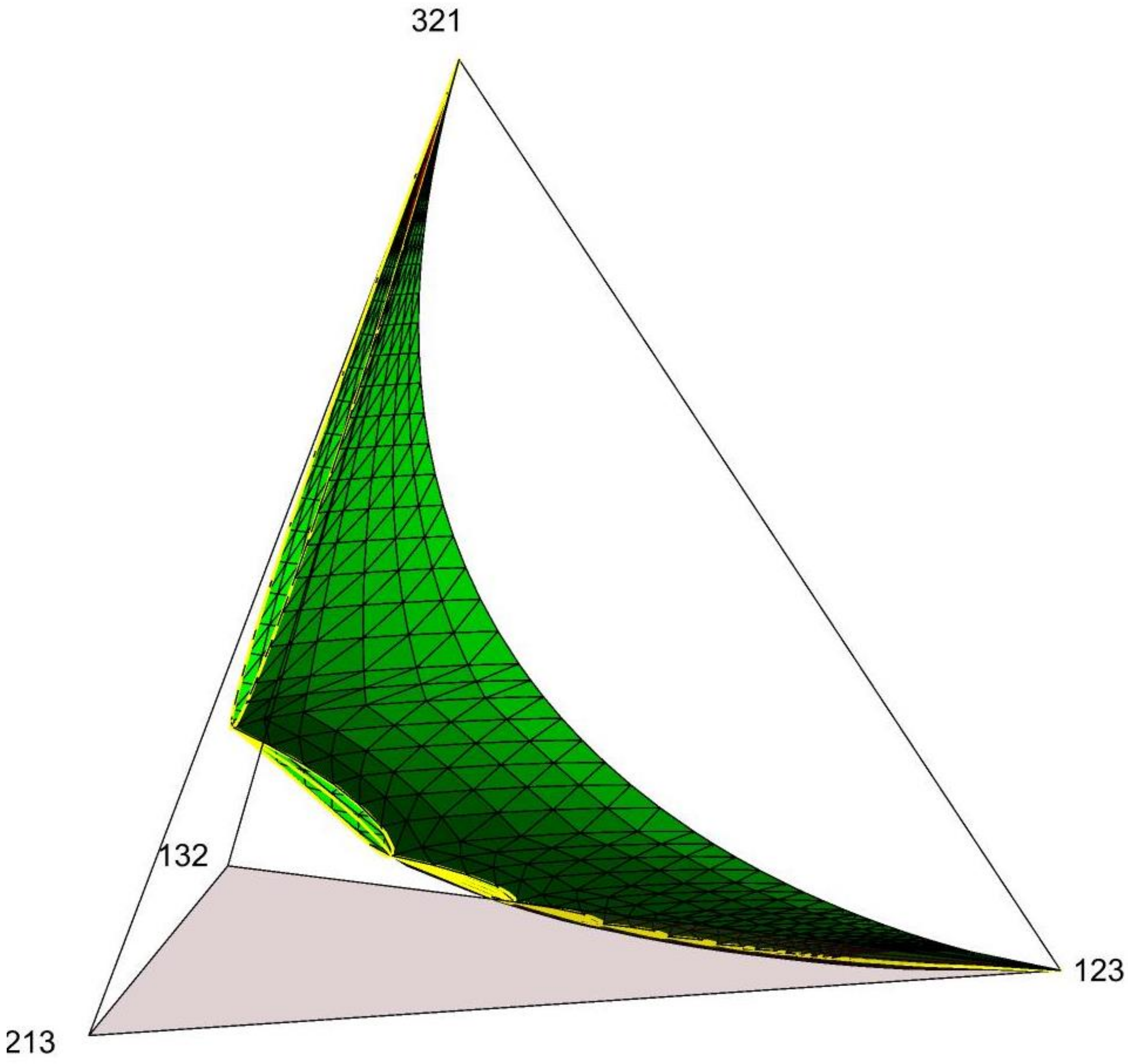
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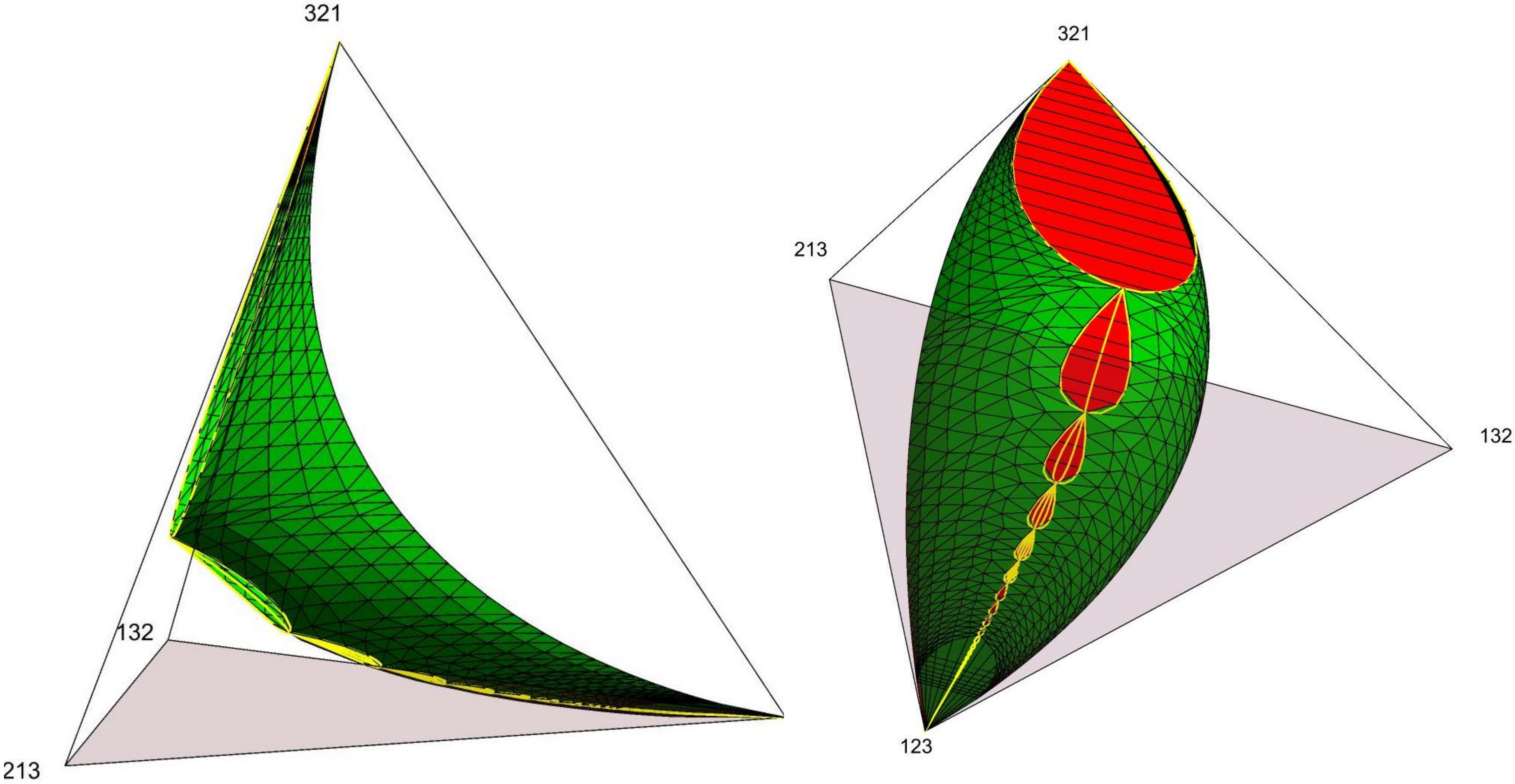
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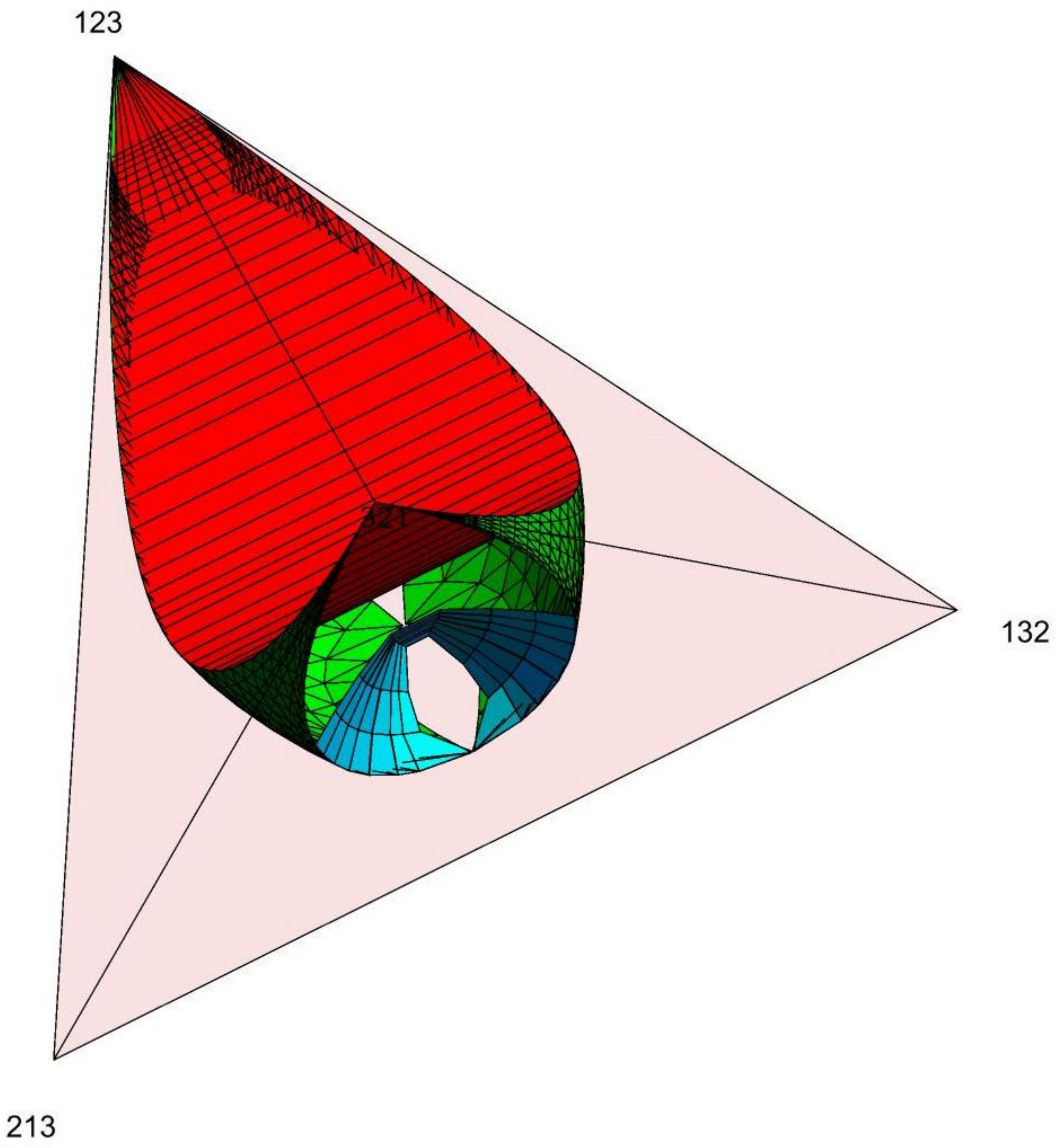
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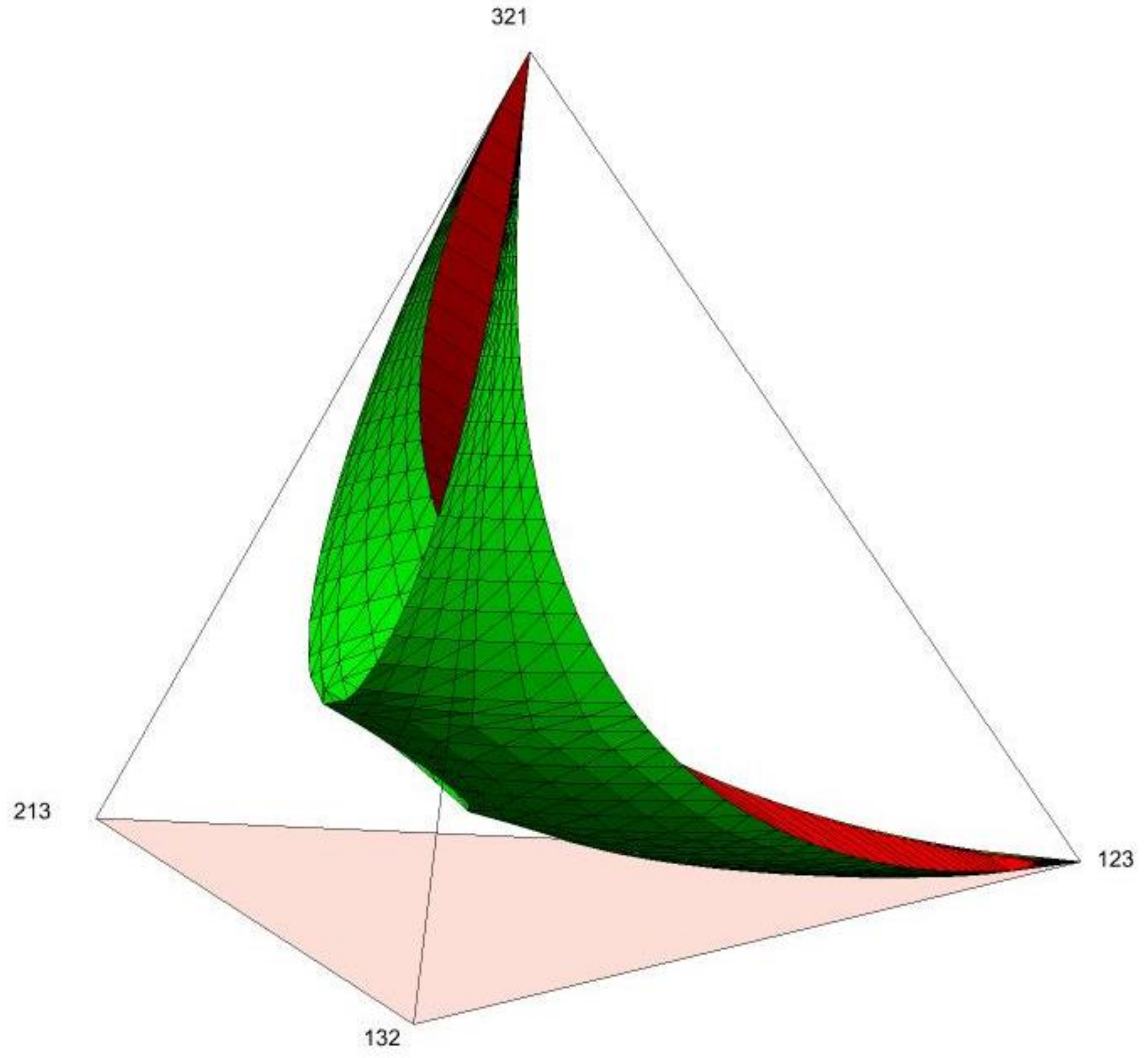
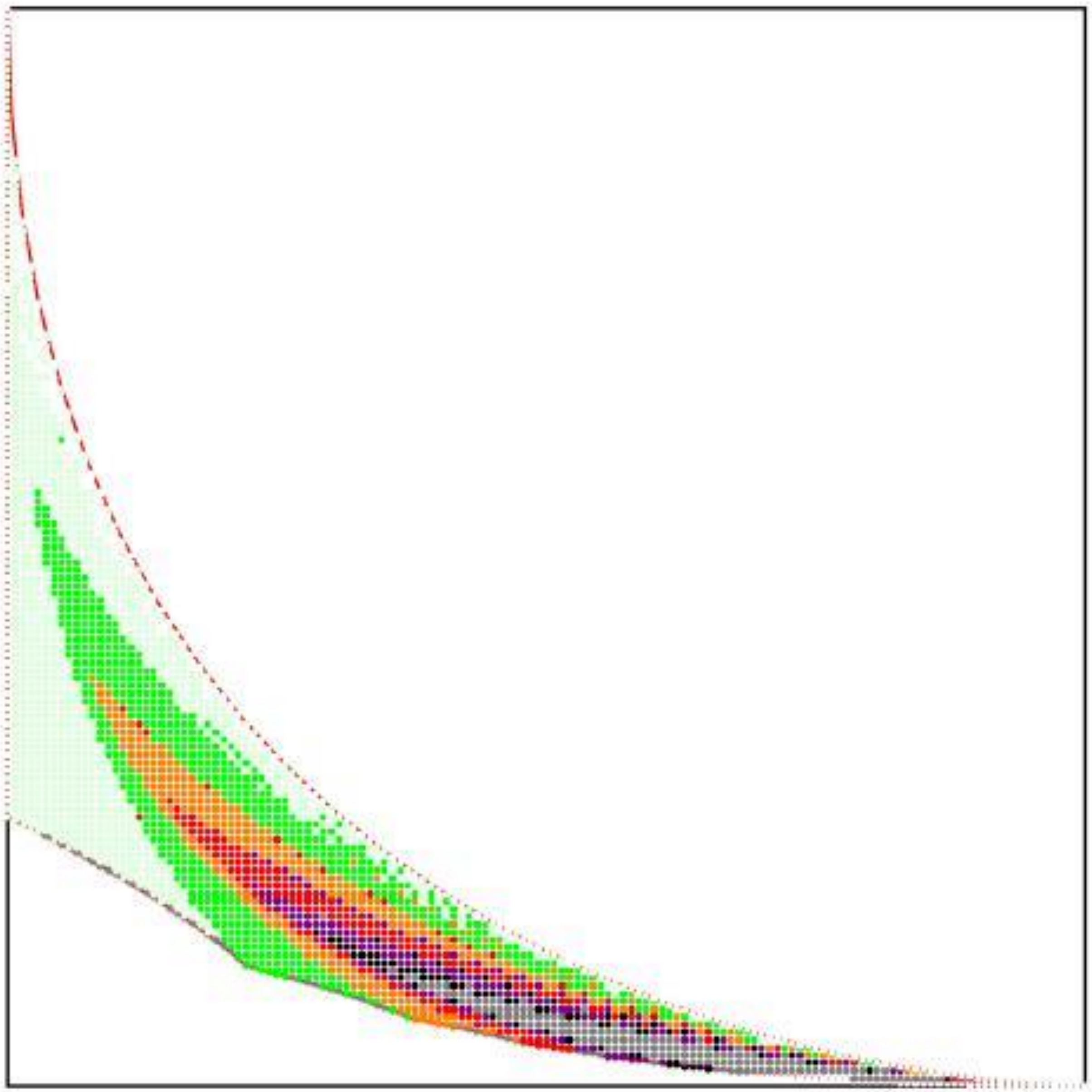
Top and Sides



How many blocks at each point of the side?



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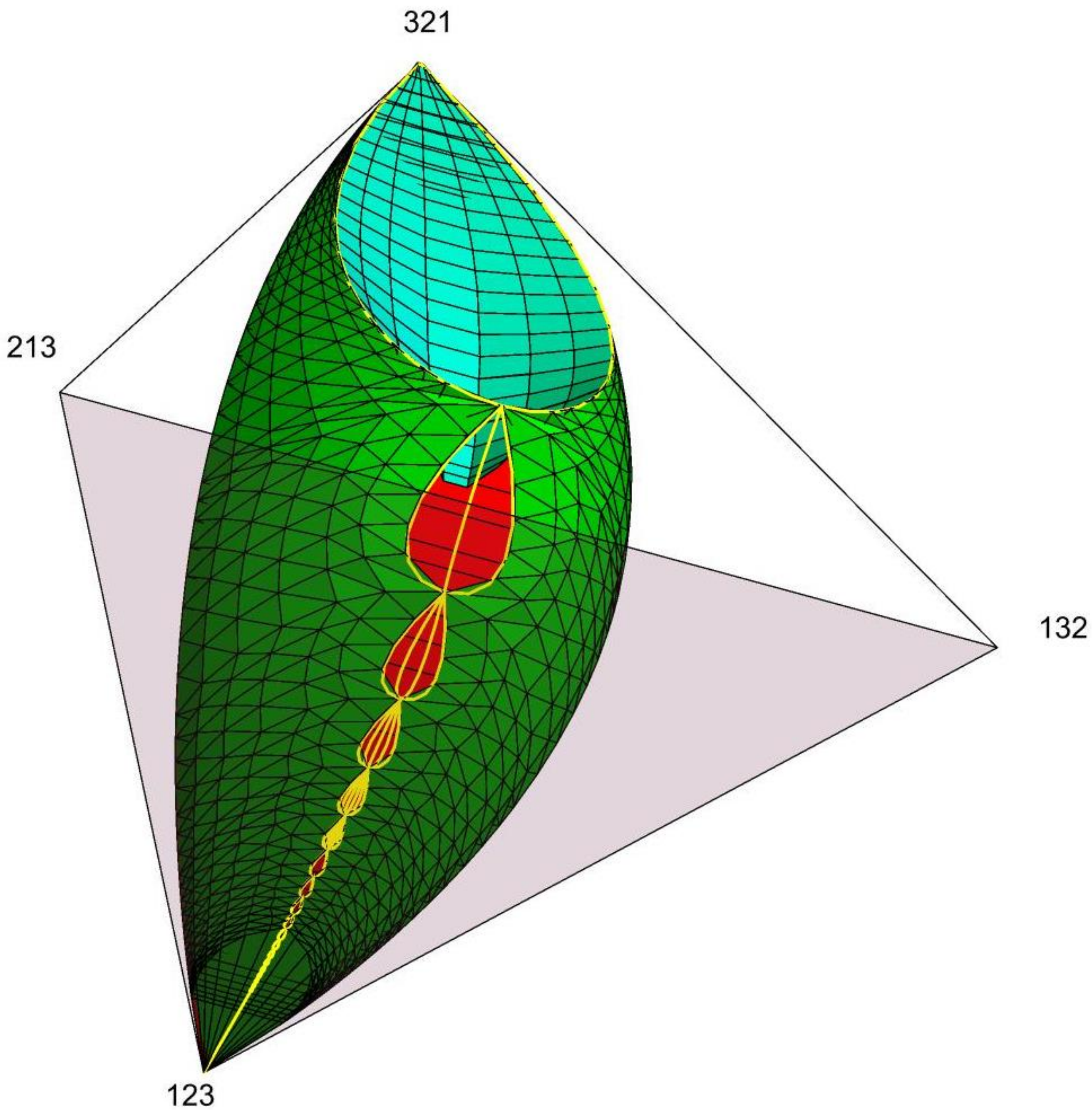
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