A Pie That Can't Be Cut Fairly

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Abstract

David Gale asked in [8, 1993] whether, when a pie is to be divided among n claimants, it is always possible to find a division that is both envy free and undominated. The pie is cut along n radii, and the claimants' preferences are described by different measures on the cake.

We answer Gale's question in the negative for $n = 3$ by exhibiting three measures for which no division of the pie can be both envy free and undominated. The measures are absolutely continuous with respect to each other and with respect to area.

1 Cutting pies

Mathematicians study cake cutting as a metaphor for more general fair division problems. Some papers on cake cutting are [7] and [11], and many more references can be found in the recent books by Brams and Taylor [5], Robertson and Webb [10], Moulin [9], and Barbanel [1]. A recent survey is by Brams [2].

A cake is cut by planes parallel to a given plane. It can therefore be represented as an interval $[0, m]$ with possible cuts corresponding to points in $[0, m]$ and possible pieces to subintervals $[a, b] \subseteq [0, m]$. We assume n players (claimants) whose preferences are represented by nonatomic measures on the interval. Letting v_i stand for the i-th player's measure, we write $v_i(S)$ or $v_i([a, b])$ for the i-th player's valuation of the piece $S = [a, b]$. We always assume that $v_i(S) = 0$ if S has length 0.

An allocation of pieces to players is called *envy free* if $v_i(S_i) \geq v_i(S_j)$ for every i and j, where S_i represents the i-th player's piece. This means that no player prefers another player's piece to its own. A player i is *satisfied* (non-envious) if $v_i(S_i) \geq v_i(S_j)$ for all j, so that an allocation is envy free precisely when all players are satisfied. An allocation $\{S_i\}$ is *dominated* by another allocation $\{T_i\}$ if $v_i(T_i) \geq v_i(S_i)$ for each i with strict inequality for at least one i. This means that the allocation $\{T_i\}$ makes at least one player better off without making any player worse off. We say that an allocation is *undominated* (or, synonymously, *efficient* or *Pareto optimal*) if it is not dominated by any other allocation.

Following [14] we call an allocation *fair* if it is both envy free and undominated. Note that both of these properties are defined without interpersonal comparisons of values.

An early result for cakes [11] is that given n players with nonatomic measures, it is always possible to find an envy-free allocation using only $n-1$ cuts. Gale showed in [8] that an envy-free allocation with $n-1$ cuts is necessarily undominated. The lesson of this cake-cutting model, then, is that fair divisions are always possible.

Pies are an alternative metaphor. Pies differ from cakes in that they are cut along radii; that is, along straight lines from the center. While we can still represent a pie by an interval $[0, m]$, we now consider the endpoints to be identified, so that the interval represents a circle that we may visualize as the circumference of the pie. A piece of pie is represented by a subinterval $[a, b]$, but we now interpret these endpoints *mod m*; that is, $a+m$ represents the same point as a. We allow the notation $[a, b]$ with $b < a$, interpreting it as $[a, b+m]$; this typically represents a piece that spans the endpoints, and would not be a single piece if it were part of a cake. The notation $[a, a]$ always represents an empty piece, while $[a, a+m]$ always represents the whole pie. We continue to represent the players' preferences by nonatomic measures v_i.

There still exist envy-free divisions. In fact, there are many more of them, because we can cut the pie at an arbitrary point and then find an envy-free division as if we were dividing a cake. But when $n \geq 3$ it is no longer the case that all envy-free divisions are undominated. Gale asked whether there must exist a fair division (*i.e.,* both envy free and undominated). Brams and Barbanel [3] showed that a fair division always exists when $n = 2$. They also found an example with $n = 4$ in which no fair division exists, but the example relies on measures that are not absolutely continuous with respect to each other—that is, some pieces have positive value to some players and zero value to others. Brams, Jones, and Klamler [4] extended the problem to the case of proportional divisions among players with unequal entitlements, and showed that in this extended problem, fair divisions might fail to exist. But for $n \geq 3$, absolutely continuous measures, and equal entitlements, Gale's question has remained open.

In this paper we answer Gale's question in the negative in the case of $n = 3$. We present a set of three measures on a pie that are absolutely continuous with each other and with respect to area, and we show that no allocation among players with these measures can be both envy free and undominated. The lesson of the pie-cutting model, therefore, is exactly opposite to the lesson of the cakes: fair divisions are not always possible.

2 The example

In all sections we represent the pie as the interval $[0, 18]$ with the endpoints identified. Points in the interval are defined *mod 18*. We call the players A, B, C, and call their measures v_A, v_B, v_C (or generically, just v_i). For describing the measures we partition the pie into 18 sectors $[0, 1]$, $[1, 2]$,. . . ,$[17, 18]$. Each measure is uniform over each segment, so to the describe the measures it suffices to list the values $v_i([k, k+1])$ for each player and each sector.

With these conventions in place, here are the measures that form the example:

	0			3			6			9			12			15		18
v_A	130	100	69	131	70	100	100	70	131	69	100	130	60	100	100	100	100	60
v_B	100	100	115	5	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	115	5	100	100	100	100	100	100	100	100

For the proof that no envy-free, undominated allocation is possible the reader may skip to section 5. The intervening sections are intended to motivate the example.

3 The failed proof

We begin with a (false) proof that an envy-free, undominated allocation always exists. We consider only the case of $n = 3$, but the proof fails just as badly for larger n.

For any allocation $\{S_A, S_B, S_C\}$ we define a *value vector* $(v_A(S_A), v_B(S_B), v_C(S_C))$. From the value vector we extract the *sum*, $s = v_A(S_A)+v_B(S_B)+v_C(S_C)$, and a *proportions vector,* $(v_A(S_A)/s, v_B(S_B)/s, v_C(S_C)/s)$.

The possible proportions vectors form a simplex as shown in Figure 1. Formally the simplex is

$$S = \{(a, b, c) \quad | \quad a, b, c \geq 0 \text{ and } a + b + c = 1 \quad \}.$$

The vertices represent allocations of the whole cake to one player. These results are easily established:

1. Every vector $(a, b, c) \in S$ is the proportions vector of an undominated allocation.

2. In every undominated allocation, at least one player is satisfied (not envious).

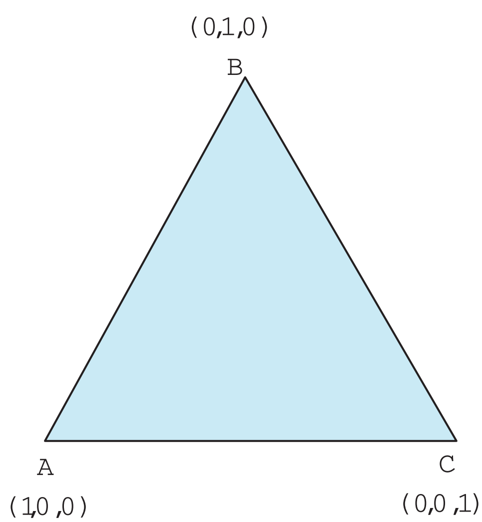


Figure 1: The simplex of proportion vectors

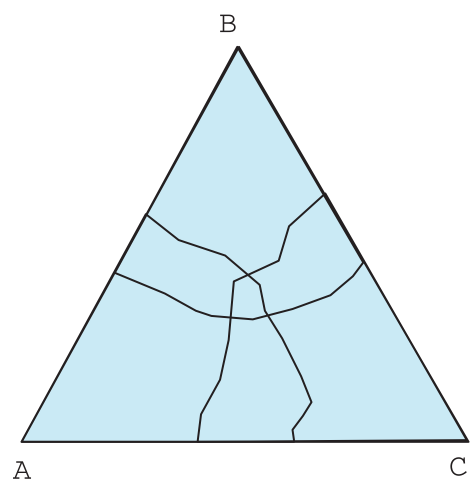


Figure 2: The sets K_A, K_B, K_C overlap

Now consider the sets

$$K_A = \{(a,b,c) \quad | \quad \text{There is an allocation with proportions vector } (a,b,c) \\ \text{in which player } A \text{ is satisfied } \}$$

and the corresponding sets K_B, K_C for players B and C. From the second result above, these sets cover S: that is, $K_A \cup K_B \cup K_C = S$. Further, K_A contains the vertex $(1,0,0)$ (because A is clearly satisfied with the whole cake) but does not intersect the side opposite $(1,0,0)$ (because in each of those allocations, A has an empty piece). Similar facts hold for K_B and K_C.

Under these circumstances, a topological lemma like the one used in [11] applies. It implies that if the K's are closed sets, they must have a point in common (Figure 2). That point might represent an undominated allocation in which all three players are satisfied. Might that imply that there is an envy-free, undominated allocation?

So goes the proof.

It has flaws. To begin with, the K's need not be closed sets. But the larger flaw is that it doesn't prove the result we want. It actually "proves" that there is a proportions vector such that every player is satisfied by *some* undominated allocation having that proportions vector. But there might be two or more undominated allocations for the same proportions vector; then A might be satisfied with one of the allocations, while B and C are satisfied with another one. There might then be no single undominated allocation that satisfies everybody.

This tells us one feature we should look for in a counterexample: there should be at least one instance of two different undominated allocations with the same proportions vector, which satisfy different sets of players.

Our strategy in building an example is to start with a case in which there is only one proportions vector with envy-free allocations. In fact, it will have two different envy-free allocations, but they will both be "brittle" in the sense that small modifications keep them from being envy free. This situation is illustrated in Figure 3. (In general, envy-free allocations are likely to be ubiquitous. We don't want that to happen in our example, because if there are many envy-free allocations, it is hard to keep them all from being undominated.)

We'll then add an overlay of additional allocations, as shown in Figure 4. The allocations in the overlay have the same proportions vectors as the old ones, and dominate them — including the old envy-free allocations.

We need to keep new envy-free allocations from arising on the overlay. We do this by making B envy C and C envy A for all allocations in the overlay. These conditions could be fixed by making A's piece smaller, but at that point we let the overlay sink below the former sheet, so that any new envy-free allocations would be dominated by allocations in the original example. The edge where the overlay intersects the original sheet is characterized by having two undominated allocations for each proportions vector.

In the next section we carry out this strategy, and build an example from the ground up.

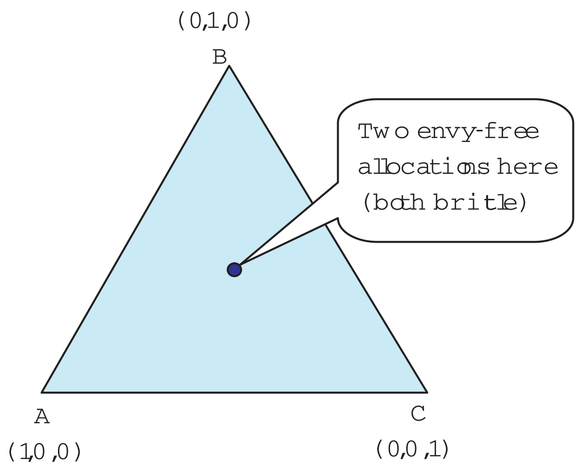


Figure 3: First layer of allocations

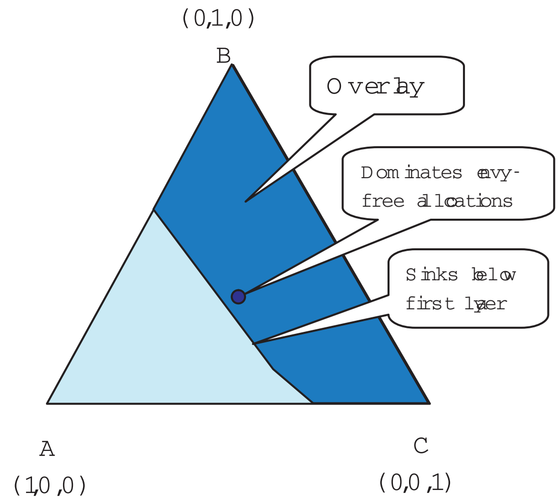


Figure 4: Overlay layer

4 Building an example

We begin the construction with these three measures:

Version 1: Two brittle envy-free allocations

	0			3			6			9			12			15		18
v_A	100	100	100	100	100	100	100	100	100	100	100	100	90	90	90	90	90	90
v_B	90	90	90	90	90	90	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	90	90	90	90	90	90	100	100	100	100	100	100

In this version, suppose that cuts are made at 0, 6, and 12. Then each player is indifferent between two pieces: A is happy with $[0, 6]$ or $[6, 12]$, B is happy with $[6, 12]$ or $[12, 18]$, and C is happy with $[12, 18]$ or $[0, 6]$. This gives us two envy-free, undominated solutions:

- Plan 1: $[0, 6]$, $[6, 12]$, $[12, 18]$ to A, B, C respectively.
- Plan 2: $[6, 12]$, $[12, 18]$, $[0, 6]$ to A, B, C respectively.

Each of these allocations delivers values of 600, 600, 600 to the respective players, which is clearly the best that can be done. But since each allocation has three players at the very edge of envy, these allocations are "brittle" as desired—small changes are likely to destroy their envy-free character.

Unfortunately, there are more envy-free allocations than these, and they are not brittle. Any allocation of the form $[x, 6+x]$, $[6+x, 12+x]$, $[12+x, x]$ with $0 \le x \le 6$ is envy-free and undominated. In order to kill off most of these allocations, we add some "decorations" to A's measure:

Version 2: Weakening some of the extra envy-free allocations

	0			3			6			9			12			15		18
v_A	$\underline{130}$	100	$\underline{70}$	100	100	100	100	100	100	$\underline{70}$	100	$\underline{130}$	90	90	90	90	90	90
v_B	90	90	90	90	90	90	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	90	90	90	90	90	90	100	100	100	100	100	100

This change causes most of the unwanted allocations to be undesirable, because A will not want to accept one of the "70" sectors without the nearby "130" sector. The other allocations can still be adjusted to make them envy-free, but they are not likely to be undominated.

Before moving on, we modify the "killer sectors" in all three measures. This change adds flexibility that we will need later.

Version 3: Adjusting the killer sectors

	0			3			6			9			12			15		18
v_A	130	100	70	100	100	100	100	100	100	70	100	130	$\underline{60}$	100	100	100	100	$\underline{60}$
v_B	100	100	100	$\underline{20}$	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	100	$\underline{20}$	100	100	100	100	100	100	100	100

Now notice that there is one extra envy-free solution that we haven't eliminated:

- Plan 3: $[3, 9]$, $[9, 15]$, $[15, 3]$ to A, B, C respectively.

We'll preserve this allocation, and use it to dominate the allocations from Plan 1 and Plan 2. To do this, we freeze it in place with another set of "decorations," and then shift 1 extra point of A's measure into each sectors $[3, 4]$ and $[8, 9]$ to make them more attractive.

Version 4: Beefing up the "Plan 3" allocation

	0			3			6			9			12			15		18
v_A	130	100	$\underline{69}$	$\underline{131}$	$\underline{70}$	100	100	$\underline{70}$	$\underline{131}$	$\underline{69}$	100	130	60	100	100	100	100	60
v_B	100	100	100	20	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	100	20	100	100	100	100	100	100	100	100

Now Plan 3 offers values of 602, 600, 600 to the respective players, so it dominates both Plan 1 and Plan 2. Note that any modification to Plan 3 that alters A's piece by more than a trivial amount will reduce its value to A below 602, which can keep it from being undominated.

But as it stands, Plan 3 is both envy free and undominated. Finally, we modify it by making B envy C and C envy A.

Version 5, final: Making players envious in Plan 3

	0			3			6			9			12			15		18
v_A	130	100	69	131	70	100	100	70	131	69	100	130	60	100	100	100	100	60
v_B	100	100	$\underline{115}$	$\underline{5}$	100	100	100	100	100	100	100	100	100	100	100	100	100	100
v_C	100	100	100	100	100	100	100	100	$\underline{115}$	$\underline{5}$	100	100	100	100	100	100	100	100

Of course these instances of envy could be corrected — but only by significant alterations of A's piece, which would cause Plan 3 to be dominated. So, the example is now complete.

Of course, we need to prove these claims.

5 The proof

In this section we proof that these measures (see Section 2, or Version 5 above) do not permit any envy-free, undominated allocation.

In outline, the proof is as follows. First, since the densities are nearly uniform, any envy-free allocation must have pieces of roughly equal length. Second, any undominated allocation with suitable piece lengths must have $v_A(S_A) + v_B(S_B) + v_C(S_C) \geq 1800$, or else it would be dominated by some other allocation (that we construct explicitly).

Next, there are only three plausible ways to achieve this total:

- Plan 1: $[0, 6]$, $[6, 12]$, $[12, 18]$, or a slight modification;
- Plan 2: $[6, 12]$, $[12, 18]$, $[0, 6]$, or a slight modification;
- Plan 3: $[3, 9]$, $[9, 15]$, $[15, 3]$, or a slight modification.

In Plan 1, some instance of envy occurs unless the pieces all have length exactly 6, in which case the allocation is dominated by $[3, 9]$,$[9, 15]$,$[15, 3]$. Plan 2 is symmetrical. In Plan 3, either B envies C or C envies A, or A's piece is reduced to the extent that the allocation is dominated.

We give the proof in greater detail.

1. In any envy-free allocation, each piece has length ℓ satisfying $5.5 < \ell < 7$.

Proof: Each player's valuation of the entire pie is $v_i([0, 18]) = 1720$. It follows that each player must value its own piece at at least 573, or it would necessarily prefer another player's piece to its own. By inspection, no piece of length $\ell \geq 5.5$ has value as great as 573 to any player, so each piece in an envy-free division must have length $\ell > 5.5$. If any piece were longer than 7, then some other piece would be too short, so we must always have $5.5 < \ell < 7$. □

2. In any allocation that is both envy free and undominated, the sum $v_A(S_A) + v_B(S_B) + v_C(S_C)$ must be at least 1800.

In general, we will refer to the quantity $v_A(S_A) + v_B(S_B) + v_C(S_C)$ as the *sum* of the allocation.

Proof: Let a, b, c satisfy $a + b + c = 1800$ and $5.5 < a, b, c < 7$. We will show that there is an allocation with $(v_A(S_A) = a, v_B(S_B), v_C(S_C)) = (a, b, c)$. In particular, the allocation is

$S_A = [0, a/100]$;
$S_B = [a/100, 18 - c/100]$;
$S_C = [18 - c/100, 18]$.

The values of v_B on S_B and of v_C on S_C are both constant at 100, so we have

$$v_C(S_C) = 100\,(18 - c/100 - a/100) = b$$

and

$$v_C(S_C) = 100\,(c/100) = c.$$

The density of v_A on S_A is not constant, but its average is clearly 100 regardless of the exact location of the right boundary of S_A. Therefore

$$v_A(S_A) = 100\,(a/100) = a$$

as required.

Any allocation that is envy-free and has sum strictly less than 1800 is dominated by one of these allocations. □

3. It follows that in any envy-free, undominated allocation, the values of the pieces must average 100 per sector, as valued by the players that receive them.

4. In an envy-free, undominated allocation, B's piece cannot include any part of the sector $[2, 3]$.

Proof: To average 100 per sector, B would have to avoid most of $[3, 4]$ so B's piece would be roughly $[14, 2+]$ or $[15, 3]$. In either case, either C or A would be stuck with most of $[9, 10]$, and the required average could not be maintained. □

5. In an envy-free, undominated allocation, C's piece cannot include any part of the sector $[8, 9]$.

Proof: If it did, then C's piece would be roughly $[3, 9]$, and A could not avoid being stuck with one of the killer cells $[12, 13]$ or $[17, 18]$. □

6. Therefore, in any envy-free, undominated solution, B's piece and C's pieces have values averaging at most 100 per sector to their respective owners. Therefore A's piece must also average at least 100 per sector.

7. By inspection, there are only three ways for A to accomplish this:

- Plan 1. $S_A = [0, 6 + x]$ where $-1 < x < 1$;
- Plan 2. $S_A = [6 + x, 12]$ where $-1 < x < 1$;
- Plan 3. $S_A = [3 + x, 9 + y]$ where $|x| + |y| \leq 2/31$.

These plans can be completed as follows:

- Plan 1. $[0, 6 + x]$, $[6 + x, 12 + y]$, $[12 + y, 18]$ to A, B, C, with $-1 < x, y < +1$;
- Plan 2. $[6 + x, 12]$, $[12, y]$, $[y, 6 + x]$ to A, B, C, with $-1 < x, y < +1$;
- Plan 3. $[3 + x, 9 + y]$, $[9 + y, 15 + z]$, $[15 + z, 3 + x]$ to A, B, C, with $|x| + |y| \leq 2/31$ and $-1 < z < +1$.

8. There is no envy-free, undominated allocation consistent with Plan 1.

Proof: The short reason is that any alteration to the boundaries causes A to envy B, or B to envy C, or C to envy A. If the boundaries are left at 0, 6, 12 exactly, the allocation is dominated by $[3, 9]$, $[9, 15]$, $[15, 3]$.

For a formal proof, we explore the consequences of envy-freeness:

- $v_A(S_A) \geq v_A(S_B)$ implies
$$200x - Ky \geq 0 \tag{1}$$
where K is either 60 or 130 depending on the sign of x.

- $v_B(S_B) \geq v_B(S_C)$ implies
$$-100x + 200y \geq 0. \tag{2}$$

- $v_C(S_C) \geq v_C(S_A)$ implies
$$-100x - 100y \geq 0. \tag{3}$$

Adding twice (1) to (3) forces $y \leq 0$ and adding (2) to twice (3) gives $x \leq 0$. Now (3) forces $x = y = z = 0$.

Therefore any envy-free allocation in Case I must be exactly $[0, 6]$, $[6, 12]$, $[12, 18]$, with the values to the players being exactly 600, 600, 600. But this allocation is dominated by $[3, 9]$, $[9, 15]$, $[15, 3]$, which delivers 602, 600, 600. □

9. There is no envy-free, undominated allocation consistent with Plan 2.

This is symmetrical to Plan 1.

10. There is no envy-free, undominated allocation consistent with Plan 3.

Proof: The problem here is that, with no adjustments to the boundaries, B envies C and C envies A. These conditions cannot both be fixed without reducing A's piece, but any sufficient reduction of A's piece causes the sum to fall below 1800.

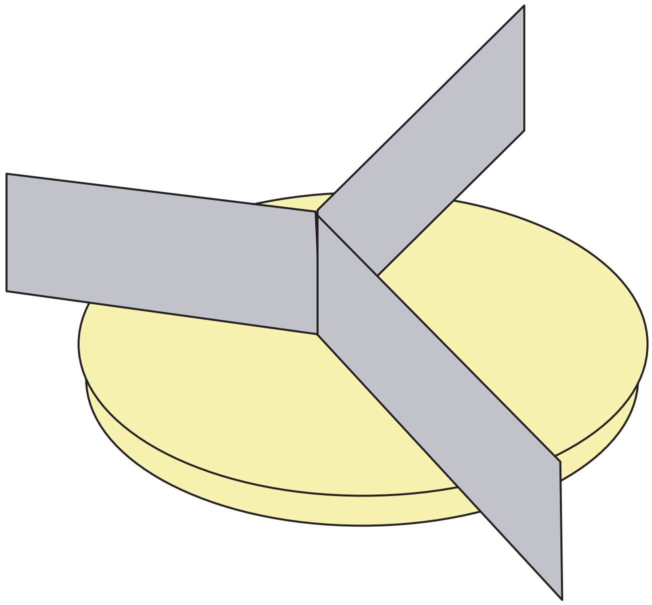
More precisely: Suppose that the left end of C's piece is at least 15 (that is, $z \geq 0$). Then C's piece is at most $[15, 3+x]$ where $x \leq 2/31$. Thus C's valuation of C's piece is at most $600+100x \leq 607$. But C's valuation of A's piece is at least $615-100x+100y \geq 608$, so C envies A.

Suppose on the other hand that $z < 0$. Then B's valuation of B's piece is at most $600 - 100y \leq 604$, while B's valuation of C's piece is at least $615 + 115x \geq 611$. So B envies C. □

We have completed the proof that these measures do not allow an envy-free, undominated allocation in any case.

6 A fresh problem

A pie-cutter has three blades, fixed at angles of 120 degrees. We can place its center anywhere we like, and orient it as we please, as long as the blades remain vertical and



their angles remain fixed. Given three players with preferences defined by absolutely continuous measures, can we use the pie-cutter to find an envy-free allocation? Can we find an allocation that is both envy free and undominated?

The blades form a Steiner tree, which suggests a generalization to more then three players. Can a pie be divided fairly among n players, with piece boundaries that form a Steiner tree? That is, that meet each other only at 120 degree angles?

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