Envy-free cake divisions cannot be found by purely finite procedures

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June 28, 2007 (first draft)

Abstract

We show that no purely finite procedure (even if unbounded) suffices reliably to find an envy-free division of a cake among three players, if only two cuts are allowed.

1 Cutting cakes

An early result for cake cutting [11] is that a cake can always be divided among n players, using $n-1$ cuts, in such a way that no player considers another piece more valuable than its own. In this model, the cake is represented by an interval, say $[0,1]$. Possible cuts are points in the interval, and possible pieces are subintervals $[x,y] \subseteq [0,1]$. Players describe their preferences by nonatomic measures on the interval. We write v_i for the i-th player's measure, so that either $v_i(P)$ or $v_i([x,y])$ represents the value of the piece $P=[x,y]$ to the i-th player.

The best current proof of this result follows Su's argument in [13]. The proof works for all $n \geq 1$. It is constructive, in the sense that an explicit sequence of possible divisions is constructed, with the actual envy-free division as its limit. Because it requires convergence, we can't call it a finite procedure.

Other non-finite procedures are available in the case of $n=3$. A moving-knife algorthm was given in [11], and a simpler moving knife algorithm is in [**?**]. Both of these procedures involve continuous operations. Further, since each of the procedures requires players to watch more than one knife, there is no obvious way of replacing the knives with, for example, a finite bidding or trimming process. So these procedures can't be called finite, either.

By contrast, there is a simple finite procedure for dividing a cake between two people—"I cut, you choose". There is also a finite procedure for envy-free division among three people, if five cuts are allowed and each player receives two of the resulting six pieces.

To start the first open problem session at the Fair Division seminar in Dagstuhl [**?**], Steven Brams asked whether there is a reliable finite procedure for envy-free cake division involving $n \geq 3$ players and $n-1$ cuts.

The purpose of this paper is to answer the question in the negative when $n=3$.

The first issue, of course, is what constitutes a finite algorithm. An answer arose from discussions at Dagstuhl, and we give a version of that answer in Section 1 by formally defining a "purely finite procedure." We do *not* require that the number of steps be bounded in advance; as a consequence, the conclusion of this paper applies to both bounded and unbounded procedures as long as they are purely finite.

In Section 2 we introduce a special class of examples which we call "rigid measure systems." A rigid measure system consists of three measures v_A, v_B, v_C with certain properties. A consequence of these properties is that there is a unique pair of cuts that produces an envy-free division among players with these measures. In this sense, the measures are rigid; but the class itself is sufficiently flexibile that the location of the cuts cannot be determined by examining a single measure, even knowing that it is part of a rigid measure system.

In Section 3 we combine these concepts to give a proof that no purely finite procedure can reliably give an envy-free division of a cake among three people, even in the special case in which we assume that the players' measures form a rigid measure system.

A short summary of the argument is this: There are many systems of measures for which there is a unique pair of cuts that produce an envy-free division. Suppose marks are made at other points. Consider the set of possible pieces formed by these marks. Then no information about the players' values of these pieces is sufficient to identify the correct cutpoints. That is because there are multiple such systems, which agree entirely at the exisiting marks and everywhere outside a neighborhood of their cutpoints, but which have different cutpoints. They can even be chosen so that one of the measures is in both systems, so that no information provided by that player can help to distinguish between the systems. This is enough to conclude that if the proper cutpoints haven't been found after a finite number of steps, then they still won't be found after one more step.

2 Purely finite procedures

This notion of purely finite procedures is a distillation of a discussion among Vangelis Markakis, Amin Saberi, Herv/'e Moulin, and Katrina Ligett, and reflects the work of Jeff Edmonds and Kirk Pruhs and a paper by Gerhard Woeginger and Jiri Sgall [14], which cites the protocol definition of Robertson and Webb [10].

We refer specifically to the case of three players and two cuts.

We say that a procedure is *purely finite* if it consists of a sequence of steps followed by the making of two cuts, subject to these conditions:

1. Some steps result in making marks on the cake, which means identifying points in $[0, 1]$. Initially the cake is marked only at its ends (that is, at 0 and 1).

2. In each step, the algorithm selects a player i, a piece $[x_1, x_2]$ where x_1 and x_2 are existing marks, a rational number q, and another existing mark described as a left endpoint y_1 or a right endpoint y_2. Then player i provides a number y_2 (if y_1 was

provided) or y_1 (if y_2 was provided) such that

$$v_i([y_1, y_2]) = qv_i([x_1, x_2]). \tag{1}$$

If no such y_2 or y_1 exists, the player so indicates. Otherwise, a mark is made at the number provided by the player.

3. After some finite number of steps, the cake is cut at two existing marks and the pieces assigned to the three players.

4. The inputs to each step are determined by the algorithm (in any way at all) from the inputs and outputs to previous steps. The same is true of the choice of when to make the cuts, where to make the cuts, and how to assign the pieces to the players.

Note that we do not require that the number of steps be bounded in advance.

The steps provided above can be combined to accomplish other finite operations. For example, we can effectively ask a player to identify a first and second choice from a list of pieces. (Cite an authority, if possible, for the proposition that every commonly-used cake-cutting operation, other than moving knives, can be accomplished by finitely many of these steps. Probably [14] is clear on that.)

We have not allowed for a separate "evaluation" operation, like the ones allowed in [14] or [10]. That is because this procedure can return an arbitrary real number, and we do not want to be concerned with the issues around describing a real number in a finite procedure. In any case, the output from an evaluation operation can only be used as the input to a subsequent cutting or marking step, and we have incorporated that usage directly.

3 Rigid Measure Systems

In this section we define rigid measure systems. A rigid measure system is a set of three players' measures with various properties. One property is that, when the players use these measures, there is only one pair of cuts that can create an envy-free division.

More precisely, let x_1, x_2, t satisfy $0 < x_1 < x_2$ and $1/3 < t < 1/2$. Then a *rigid measure system* (RMS) with *parameters* x_1, x_2, t is a set of three measures $\{v_i\}$ for $i = 1, 2, 3$ with these properties:

1. The density of each measure is always strictly between $M^- = 2^{-1/4} \approx 0.84$ and $M^+ = 2^{+1/4} \approx 1.19$. Equivalently: If P is a piece of length ℓ, then

$$M^-\ell < v_i(P) < M^+\ell$$

for every i.

2. The player's values for pieces defined by x_1 and x_2 are given by the following table. (In the table, and for the rest of this paper, s always stands for $1 - 2t$. Thus, s and t are positive numbers with $s < t$ and $s + 2t = 1$.)

	$[0, x_1]$	$[x_1, x_2]$	$[x_2, 1]$
v_A	t	t	s
v_B	s	t	t
v_C	t	s	t

We prove some lemmas about rigid measure systems.

Lemma 1 *The vectors (x_1, x_2, t) for which rigid measure systems exist form an open set in $\mathbf{R}^3$.*

□

The next lemma says that players in an RMS cannot differ too extremely in their comparison of two pieces.

Lemma 2 *Let i and j be two players with measures in a rigid measure system, and let P_1 and P_2 be any two pieces. If player i values P_1 as at least twice the value of P_2, then player j values P_1 as more than half the value of P_2. That is: If*

$$v_i(P_1) \geq 2v_i(P_2)$$

then

$$v_j(P_1) > \frac{1}{2}v_j(P_2).$$

Proof: Suppose P_1 and P_2 have lengths ℓ_1 and ℓ_2 respectively. Then

$$\begin{aligned} v_j(P_1) &> M^-\ell_1 \\ &= (\frac{M^-}{M^+})M^+\ell_1 \\ &> (\frac{M^-}{M^+})v_i(P_1) \\ &\geq 2(\frac{M^-}{M^+})v_i(P_2) \\ &> 2(\frac{M^-}{M^+})M^-\ell_2 \\ &= 2(\frac{M^-}{M^+})^2M^+\ell_2 \\ &> 2(\frac{M^-}{M^+})^2v_j(P_2) \\ &= \frac{1}{2}v_j(P_2), \end{aligned}$$

because $\frac{M^-}{M^+} = 2^{-1}$. (OK, it's really $2^{-1/2}$. That's good enough for the inequality, but we really ought to get the exponents right in the definition of M^+ and M^-. That's why this is a draft.) □

The next lemma is the reason for rigid measure systems.

Lemma 3 *If v_A, v_B, and v_C form a rigid measure system with parameters x_1, x_2, t, then every two-cut, envy-free division of the cake among A, B, C has its cuts at x_1 and x_2.*

Proof: The cuts x_1 and x_2 are unique, but here are always two envy-free ways to distribute the pieces to the players. From the left to right, the pieces can be given to A, B, C or to C, A, B.

Suppose that instead of making cuts at x_1 and x_2, we made them at y_1 and y_2.

If $y_1 \leq x_1$ and $y_2 \geq x_2$ with strict inequality in at least one case, then both A and B would insist on the middle piece. They would perceive its value to be strictly greater than t, and other pieces' value to be t or less.

If $y_1 \geq x_1$ and $y_2 \leq x_2$ with strict inequality in at least one case, then no player would accept the middle piece. Each player would consider some other piece to have value at least t, and the middle piece to have value strictly less than t.

By symmetry we are left only with the case $y_1 \geq x_1$ and $y_2 \geq x_2$ with strict inequality in at least one case. It is clear that neither A nor C will accept the rightmost piece, as they consider the leftmost piece more valuable. So if the division is envy free the rightmost piece must be accepted by B, who must consider it at least the equal of the middle piece. We will show that A and C must then both insist on the leftmost piece.

Let $P_1 = [x_1, y_1]$ and $P_2 = [x_2, y_2]$. If B accepts the rightmost piece, it is because

$$\begin{aligned} v_B([y_2, 1]) &\geq v_B([y_1, y_2]) \\ v_B([x_2, 1]) - v_B(P_2) &\geq v_B([x_1, x_2]) + v_B(P_2) - v_B(P_1) \\ v_B(P_1) &\geq 2v_B(P_2). \end{aligned}$$

So B considers P_1 to be twice as valuable as P_2. So the other players must consider P_1 to be more than half as valuable as P_2. By a calculation like the one just made, it follows that

$$v_A([0, y_1]) > v_A([y_1, y_2])$$

so player A insists on the leftmost piece. The same is true of player C by a wider margin. Therefore no player will accept the middle piece, and the division with y_1 and y_2 is not, in fact, envy free. This completes the proof of the lemma. □

The next lemma tells us that one player in a rigid measure system can't determine the parameters of the system from his own measure. For example, suppose that the player A's measure is v_A, and that the actual parameters of the system include x_1 and x_2. Then there are other rigid measure systems that also include v_A as one of the measures, but which have parameters $y_1 \neq x_1$ and $y_2 \neq x_2$. Further, these other systems can be chosen so that they agree with the original system for any piece that does not start or end near x_1 or x_2.

Lemma 4 *Let v_A, v_B, and v_C form a rigid measure system with parameters x_1, x_2, t. Let $\epsilon > 0$. Then there exists a rigid measure system v_A, v'_B, v'_C—that is, consisting of the original v_A and two new measures v'_B and v'_C—such that*

- *The new measures agree with the old measures outside a neighborhood of x_1, x_2. Specifically, if $[x, y]$ is any piece such that neither x nor y is within ϵ of x_1 or x_2, then $v'_B([x, y]) = v_B([x, y])$ and $v'_C([x, y]) = v_C([x, y])$.*

- *The parameters of the new system are y_1, y_2, t' with $y_1 \neq x_1$ and $y_2 \neq x_2$.*

Proof: Omitted from this draft. □

4 There is no finite procedure

Theorem 1 *There is no purely finite procedure that reliably finds an envy-free division of a cake.*

Proof: We show that the theorem is true even if the players' measures form an RMS, and even if the algorithm is allowed to assume that fact.

To do this, suppose that the parameters of the RMS are x_1, x_2, t. We show by induction on N that after N steps, no algorithm can reliably place a mark at either x_1 or x_2.

Suppose, indeed, that after $N - 1$ steps, no mark has been placed at x_1 and x_2. Let ϵ be the distance from x_1 or x_2 to the nearest mark to either of these points.

Suppose that player A is selected for step N.

The existing marks are consistent with some RMS with the parameters given. But by the last lemma, they are also consistent with some other RMS involving player A's measure, but different parameters y_1 and y_2. No question addressed to player A can produce an answer that is equal to one of x_1 and x_2, and also equal to one of y_1 and y_2. So, whatever question is selected by the algorithm, it cannot reliably leave a mark at an appropriate cutpoint after N steps.

□

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