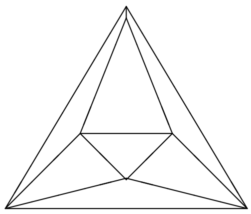
Test 3 — Answers

Week 2, Day 4 — Friday, July 13, 2012 test answers (WRS)

1. Draw the graph $K_{2,2,2}$ in a plane, with no edge-crossings. (If it is impossible, prove that it is impossible.)

   (The graph $K_{2,2,2}$ has six vertices in three pairs—let's call them u and v, w and x, y and z. Every edge between two vertices is present **except** the edges uv, wx, yz. That means that there are 12 edges in all.)

   Solution. It's the graph of the edges of an octahedron.



2. The entries of an $n \times n$ table are **non-negative** reals such that the numbers in each row add to 1, and the numbers in each column add to 1. Prove that one can pick n **positive** numbers, including exactly one from each row and exactly one from each column.

   (The example below isn't very hard by itself, but it shows what is going on.)

1/3	1/3	1/3	0
2/3	0	1/3	0
0	2/3	0	1/3
0	0	1/3	2/3

1/3	1/3	[1/3]	0
[2/3]	0	1/3	0
0	[2/3]	0	1/3
0	0	1/3	[2/3]

   Solution. It is always possible. If it were impossible, then by the marriage theorem (Hall's theorem) there would be $n-1$ lines, horizontal and/or vertical, covering all of the positive entries. But then the sum along all of the lines, even with possible double counting, would be just $n-1$ (since each line has sum 1), and that isn't enough to cover all of the positive entries in the table, which have sum n.

3. (a) Construct a graph G with 18 vertices and 81 edges, which has no 3-cycles.
   (b) Show that if a graph H has 18 vertices and 82 edges, then it contains a 3-cycle.

   Solution to part a. The complete bipartite graph, $K_{9,9}$, works.

   Solution to part b. Suppose that there is no 3-cycle. We will argue to a contradiction.

   The sum of the vertex degrees is $2 \cdot 82 = 184$, so the average vertex degree is strictly greater than 9, so there must be a vertex with at least 10 neighbors. We'll consider the case of a vertex v with exactly 10 neighbors. (The cases of 11 or more neighbors require attention, but they work the same way and are easier.) That leaves 7 vertices that are not v or neighbors of v.

   Note that there can be no edges between neighbors of v, because any such edge would create a 3-cycle.

   Is there an edge between any two of the 7 extra vertices? **If not,** then there can be only 80 edges at most—10 from v to its neighbors and 70 between the "10" and the "7"—which contradicts the hypothesis that there are 82 edges.

   If there are edges among the 7 it just makes it worse. If there is an edge between w and x (among the 7) then no vertex among the 10 can be connected to both w and x. If there is just one edge among the 7 that means that there can be at most $10 + 60 + 1 = 71$ edges in all, and the other cases go by very quickly. In no case is there room for the supposed 82 edges.

   (One can also use induction to show that in any graph with $2n$ edges and at least $n^2 + 1$ vertices, there must be a 3-cycle. If you are trying to avoid 3-cycles, the bipartite graph is the best you can do.)

   Better solution to part b (added after first printing). Consider any graph of size $2n$ with no 3-cycles. Let d be the largest vertex degree in the graph, let v be a vertex of degree d, and let S be the set of neighbors of v, and let T be the set of vertices not in S (so that v itself is in T). There are no edges within S, as that would make a 3-cycle, so every edge touches T. There are $2n - d$ vertices in T and each has degree at most d, so there are at most $(2n - d)d$ edges in the graph. (Double-counting some edges doesn't change this inequality.) Varying d, the quantity $(2n - d)d$ is never greater than n^2, so the graph cannot have more than n^2 edges.

4. Given a positive integer n, let $f(n)$ be the largest number of regions in the plane that can be formed by n circles. (The circles don't need to be the same size. Don't forget to count the "outside" region. Examples: $f(1) = 2$, $f(2) = 4$, $f(3) = 8$.) Write a formula for $f(n)$.

   Solution. When we add the nth circle we can cause it to intersect each previous circle twice, for a total of $2(n - 1)$ intersections. So the new circle passes through $2(n-1)$ of the existing regions, dividing each of them in two, and that adds $2(n-1)$ to the number of regions. So we have the recursion

$$f(n) = f(n - 1) + 2(n - 1)$$

which has the solution (it's an arithmetic series)

$$f(n) = 2 + (n^2 - n).$$

(That last formula was corrected after the first printing.)

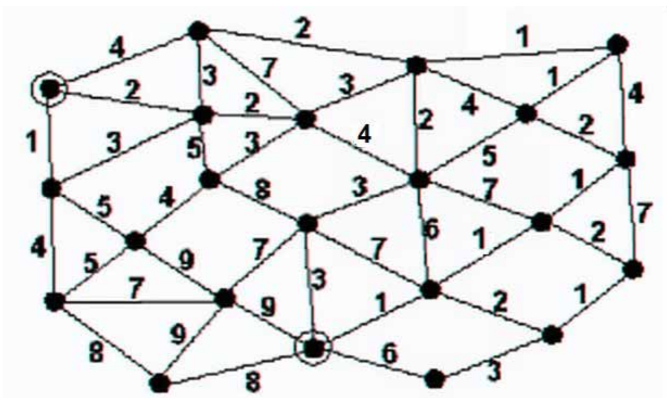
5. Given a permutation of length n, a **descent** is a pair of consecutive entries with decreasing values, and an **ascent** is a pair of consecutive entries with increasing values. More precisely, the permutation σ has a descent ending at position i if $\sigma(i-1) > \sigma(i)$, and an ascent ending at position i if $\sigma(i-1) < \sigma(i)$. Example: If $\sigma = 234165$, then σ has two descents, ending at the values 1, 5; and three ascents, ending at the values 3, 4, 6.

   Show by any method that the number of permutations of size n with 3 ascents is the same as the number of permutations of size n with 3 descents.

   Solution. Make a list of the permutations of size n with 3 descents. Write them each backwards. Now you have a list of the permutations of size n with 3 ascents.

   (The number of such permutations is an Eulerian number, which we called $A(n, 3)$.)

6. Consider the following graph, with costs shown on each edge.



   (There are separate pages available with extra copies of this graph, if you would like to use them for scratch work. You can turn them in if you like, but it is enough to turn in only the numerical answers.)

   (a) Find the cost of the least-cost path between the two circled vertices.

   (b) Find the cost of the minimal-cost spanning tree.

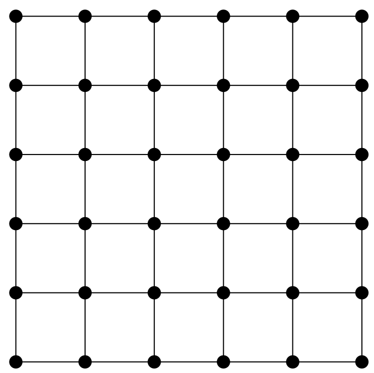
   Solution. (a) 13; (b) 54.

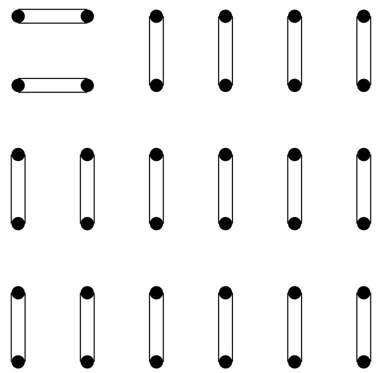
7. (a) In how many ways can a set of 36 points be partitioned into pairs?

   (b) Here is a graph with 36 vertices (left picture, below). The edges are the segments of length 1 between adjacent vertices. Can you select 18 edges such that (a) your 18 edges include all 36 vertices and (b) exactly one of the edges is horizontal, the rest being vertical?

   (c) In the same graph, can you select 18 edges such that (a) your 18 edges include all 36 vertices and (b) exactly 9 of the edges are horizontal, the rest being vertical?

   (In the picture on the right below, there is a nice example with 2 horizontal edges.)





Solution to part a. 35!!, or $\dfrac{36!}{(2^{18})(18!)}$.

Solution to part b. No, it's impossible. Your horizontal edge would leave an odd number of vertices in one of the columns—specifically, it would leave 5 vertices. (Actually this would happen in two columns.) Those 5 could only be connected to each other by verticals, and that's impossible.

Solution to part c. No, it's still impossible. Color the odd-numbered columns red and the even-numbered columns blue. Your horizontal edges would account for 9 vertices in red columns, and 9 vertices in blue columns. That would leave 9 red-column vertices and 9 blue-column vertices. Somewhere there would be a column with an odd number of vertices remaining to be covered, and you would not be able to cover them with vertical edges.

(end of test)