Test 3

Week 2, Day 4 — Friday, July 13, 2012 (WRS)

1. Draw the graph $K_{2,2,2}$ in a plane, with no edge-crossings. (If it is impossible, prove that it is impossible.)

   (The graph $K_{2,2,2}$ has six vertices in three pairs—let's call them u and v, w and x, y and z. Every edge between two vertices is present **except** the edges uv, wx, yz. That means that there are 12 edges in all.)

2. The entries of an $n \times n$ table are **non-negative** reals such that the numbers in each row add to 1, and the numbers in each column add to 1. Prove that one can pick n **positive** numbers, including exactly one from each row and exactly one from each column.

   (The example below isn't very hard by itself, but it shows what is going on.)

1/3	1/3	1/3	0
2/3	0	1/3	0
0	2/3	0	1/3
0	0	1/3	2/3

1/3	1/3	[1/3]	0
[2/3]	0	1/3	0
0	[2/3]	0	1/3
0	0	1/3	[2/3]

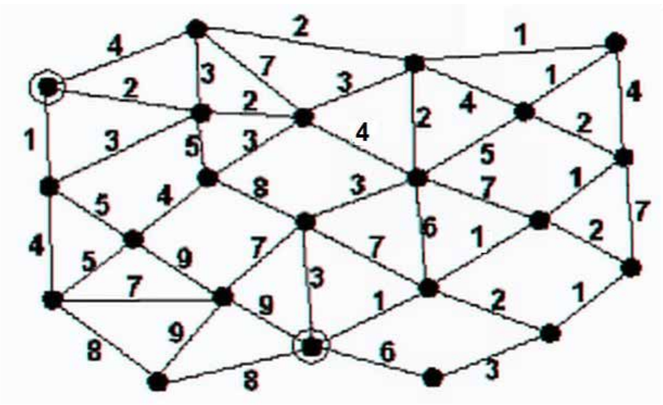
3. (a) Construct a graph G with 18 vertices and 81 edges, which has no 3-cycles.
   (b) Show that if a graph H has 18 vertices and 82 edges, then it contains a 3-cycle.

4. Given a positive integer n, let $f(n)$ be the largest number of regions in the plane that can be formed by n circles. (The circles don't need to be the same size. Don't forget to count the "outside" region. Examples: $f(1) = 2$, $f(2) = 4$, $f(3) = 8$.) Write a formula for $f(n)$.

5. Given a permutation of length n, a **descent** is a pair of consecutive entries with decreasing values, and an **ascent** is a pair of consecutive entries with increasing values. More precisely, the permutation σ has a descent ending at position i if $\sigma(i-1) > \sigma(i)$, and an ascent ending at position i if $\sigma(i-1) < \sigma(i)$. Example: If $\sigma = 234165$, then σ has two descents, ending at the values 1, 5; and three ascents, ending at the values 3, 4, 6.

   Show by any method that the number of permutations of size n with 3 ascents is the same as the number of permutations of size n with 3 descents.

6. Consider the following graph, with costs shown on each edge.



(There are separate pages available with extra copies of this graph, if you would like to use them for scratch work. You can turn them in if you like, but it is enough to turn in only the numerical answers.)

(a) Find the cost of the least-cost path between the two circled vertices.

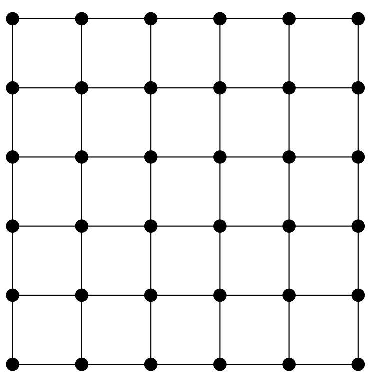
(b) Find the cost of the minimal-cost spanning tree.

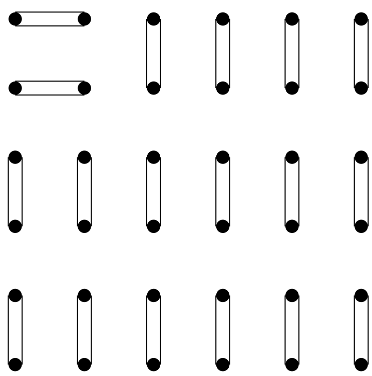
7. (a) In how many ways can a set of 36 points be partitioned into pairs?

(b) Here is a graph with 36 vertices (left picture, below). The edges are the segments of length 1 between adjacent vertices. Can you select 18 edges such that (a) your 18 edges include all 36 vertices and (b) exactly one of the edges is horizontal, the rest being vertical?

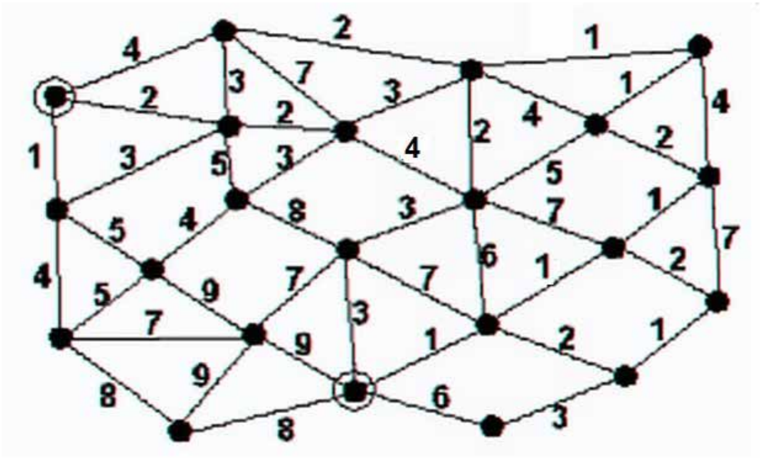
(c) In the same graph, can you select 18 edges such that (a) your 18 edges include all 36 vertices and (b) exactly 9 of the edges are horizontal, the rest being vertical?

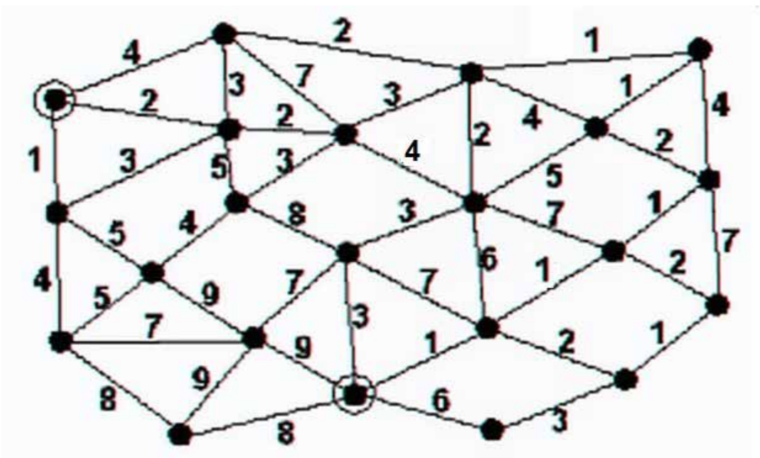
(In the picture on the right below, there is a nice example with 2 horizontal edges.)

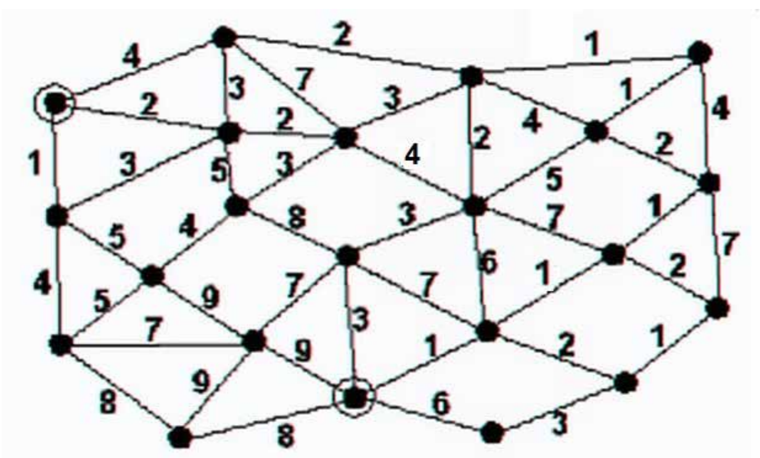




(end of test)


2 1
4 7 3 1 4
2 3 2 4 2 4
1 3 5 3 4 5 2
5 4 8 3 7 1 7
4 5 9 7 7 6 1 2
7 3 1 9 1 2
8 9 8 6 3


2 1
4 7 3 1 4
2 3 2 4 2 4
1 3 5 3 4 5 2
5 4 8 3 7 1 7
4 5 9 7 7 6 1 2
7 3 1 9 1 2
8 9 8 6 3


2 1
4 7 3 1 4
2 3 2 4 2 4
1 3 5 3 4 5 2
5 4 8 3 7 1 7
4 5 9 7 7 6 1 2
7 3 1 9 1 2
8 9 8 6 3