Problem Set 13—Special Invariant Problems

Week 3, Day 2 — Wednesday, July 11, 2012 (extra) (WRS)

1. All squares in a rectangular 6×6 board are labeled by the number 0, except that the upper-right corner and the lower-left corner are labeled 1. At any time we can pick two adjacent squares (squares with a common edge) and increase each of their labels by the same amount. Can we make all of the labels equal?
2. Two hooligans enjoy breaking glass. Hooligan S can break any piece of glass into 7 pieces, and hooligan T can break any piece of glass into 10 pieces. Together, can they break one piece of glass into 2012 pieces?
3. Three grasshoppers are positioned in the plane at 3 corners of a square. A grasshopper at point A can jump over a grasshopper at point B, and land at the point C which is symmetric to A with respect to B. (As vectors, $C = 2B - A$.) Can any grasshopper get to the fourth corner of the original square?
4. Three grasshoppers A, B, C are in a line. At each second, the left one jumps over the other two. Can they be in the same order (A, B, C) after 2012 seconds?
5. Three grasshoppers A, B, C are in a line. At each second, the middle one jumps over one of the other two. Can they be in the same order (A, B, C) after 2012 seconds?
6. In a rectangular array of numbers, we are allowed to change the sign of every entry in a row, or to change the sign of every entry in a column. Can we make the sums in every row and in every column nonnegative?
7. Every member of Congress has no more than 3 enemies in Congress. Can we partition the members into two committees in such a way that nobody has more than one enemy in his or her committee?
8. Consider this sequence of ten numbers: $1, 0, 0, 0, 0, 0, 0, 0, 0, 0$. At each step we pick two numbers, not necessarily consecutive, and replace them with their arithmetic mean. What is the smallest number that can ever occupy the first position?
9. One of the corner squares in a 3×3 grid is black, and the others are white. In one move we can change all of the colors in a row (white to black or black to white), or change all of the colors in a column. Can we make all of the squares be the same color?
10. Four numbers are written around the outside of a circle. We draw another circle outside of the numbers, and between any two of the original numbers, we write their absolute difference outside of the new circle. (Given x and y, we write $|x-y|$). Then we repeat the process. Show either (a) eventually, we reach a circle of all zeros or (b) we don't necessarily ever reach a circle of all zeros.

(end)